Independence of Multiple Events

Events $A_1, A_2, ..., A_n$ are independent if:

$$P\left( \bigcap_{i \in S} A_i \right) = \prod_{i \in S} P(A_i) \quad \text{for every subset } S \subseteq \{1, 2, 3, ..., n\}.$$ 

Example 3 events $A_1, A_2, A_3$:

What are all possible subsets of $\{1, 2, 3\}$:

- $\{1, 2\}$, $\{1, 3\}$, $\{2, 3\}$, $\{1, 2, 3\}$

Then:

$$\begin{align*}
P(A_1 A_2) &= P(A_1) P(A_2) \\
P(A_1 A_3) &= P(A_1) P(A_3) \\
P(A_2 A_3) &= P(A_2) P(A_3)
\end{align*}$$

Pairswise independent.
P(A_1A_2A_3) = P(A_1)P(A_2)P(A_3)

for us to call A_1, A_2, A_3 indp.

Ex 2 indp. fair coin tosses.

H_1: \{ 1st toss is heads\}
H_2: \{ 2nd toss is heads\}

\[ H = \{ 2 \text{ tosses have different results} \} \]
\[ \Rightarrow P(H) = \frac{1}{4} = \frac{1}{2} \]

\( D = \{ \text{2 tosses have different results} \} \)

\( H_1, H_2 \text{ and } D \text{ indp?} \)

\( H_1 \times H_2 \text{ are indp.} \)
\[ P(H_1H_2) = P(H_1)P(H_2) \]

\( H_1 \text{ and } D \text{ indp?} \)
\[ P(CD | H_1) = \frac{P(H_1D)}{P(H_1)} = \frac{1}{4} = \frac{1}{2} = P(CD) \]
\[ \Rightarrow D \times H_1 \text{ are also indp.} \]
How about $P(H_1, H_2, D) \neq P(H_1) P(H_2) P(D)$?

$P$ not True $\neq 0$

$\Rightarrow$ $H_1, H_2,$ and $D$ are not indep.
Independent Trials

Seq of independent, but identical stages in an experiment.

If 2 outcomes \rightarrow Bernoulli Trials.

Example: n independent tosses of a coin.
What is the probability of getting $k$ heads in a sequence of $n$ trials?
for one seq of k heads and n-k tails
\[ p(\cdot) = p^k (1-p)^{n-k} \]

How many such seq are there?

\[ \binom{n}{k} = \frac{n!}{k! (n-k)!} = \binom{n}{n-k} \]

\( n \) choose \( n-k \)

Ex. \( n=3 \), \( k=2 \)
\[ \frac{3!}{2! 1!} = 3 \]

\[ p(k \text{ heads}) = \binom{n}{k} p^k (1-p)^{n-k} \]
\[ = \frac{n!}{k! (n-k)!} p^k (1-p)^{n-k} \]
\[ \sum_{k=0}^{n} P(k \text{ heads}) = 1 \Rightarrow \sum_{k=0}^{n} \frac{n!}{k! (n-k)!} p^k (1-p)^{n-k} = 1 \]

For a binomial distribution, if \( n = 100 \), the probability mass function gives the probability of getting exactly \( k \) heads in \( n \) trials.
Q: What is prob

\[ \frac{1,000,000,000}{4,000} = 250,000 \]

\[ \frac{25}{3 \times 10^{-6}} \]

Ex

\[ C \ll n \]

Each customer dials up with prob (p) indep. of the others.

Q: What is prob that customers get rejected?
Define $p(k) = \Pr(\text{k subs are correct})$

\[
p(\cdot) = \sum_{k=C+1}^{n} p(k) = C = p(C+1) + p(C+2)
\]

\[
p(k) = \binom{n}{k} p^k (1-p)^{n-k}
\]

\[
p(\cdot) = \sum_{k=C+1}^{n} \binom{n}{k} p^k (1-p)^{n-k}
\]

\[
\text{Ex: } n=100, \quad p=0.1, \quad C=15
\]

\[
\Rightarrow p(\cdot) = 0.0399
\]
Conducting a PRD helps us determine possible results in stage 2. If all possible results at stage 2 are possible results at stage 3, the total number of possibilities is $n_1 \cdot n_2 \cdot n_3 \ldots = n_1 \cdot n_2 \cdot n_3$. 

1. Process possible results in stage 2.
2. For ALL possible results at stage 2, there are $n_1$ possible results at stage 3.
Example

Three digit phone:

\[ 10 \times 10 \times 10 \times 10 \times 10 \times 10 = 10^7 \]

If no 0 or 2 in 1st digit:

\[ \frac{1}{8} \times 10 \times 10 \times 10 \times 10 \times 10 \times 10 = 8 \times 10^6 \]

Selecting \( k \) objects out of \( n \)

- Order matters: permutation
- Order doesn't matter: combination
Choose 2 out of 4 balls, how many different sequences can I get?

Order matters.

4 \times 3 = 12
n objects choose k

1. First time 1 out of n left with n-1 objects.
2. 2nd time 1 out of n-1
3. 3rd 1 out of n-2

kth

\[
\frac{n \times (n-1) \times (n-2) \times \cdots \times (n-k)}{(n-k) \times (n-k-1) \times (n-k-2) \times \cdots \times 2 \times 1}
\] # of permutations
\[ \frac{n!}{(n-k)!} \]

- \( n \) objects, choose \( k \).

**Example:** How many distinct words are there.

\[ \frac{26!}{22!} = 26 \times 25 \times 24 \times 23 \]

\[ \approx 358,000 \]