Examples of mean & Variance

Ex 1 Bernoulli ; flip of coin.

$$P_x(k) = \begin{cases} p & k=1 \to \text{heads} \\ 1-p & k=0 \to \text{Tails.} \end{cases}$$

$$\text{mean} = \sum_{x_0} p_x(x_0)\, x_0 = p \cdot 1 + (1-p) \cdot 0 = $$
$$= p.$$

$$\text{Var} = E[x^2] - E^2(x)$$

$$E[x^2] = \sum_{x_0} x_0^2\, p_x(x_0) = 1^2 \cdot p + 0^2 \cdot (1-p)$$
$$= p.$$

$$\text{Var} = p - p^2 = p(1-p)$$

Ex fair die 6 sided.

$$P_X(k) = \begin{cases} 1/6 & k = 1,2,3,4,5,6 \\ 0 & \text{otherwise.} \end{cases}$$

$$E[X] = \sum_{x_0} x_0 \, p_X(x_0) = \frac{1}{6}\cdot 1 + \frac{1}{6}\cdot 2 + \frac{1}{6}\cdot 3 + \cdots + \frac{1}{6}\cdot 6$$

$$= 3.5$$

$$Var[X] = E(X^2) - E^2(X)$$

$$E[X^2] = \sum_{x_0} x_0^2 \, p_X(x_0) = \frac{1}{6}\cdot 1^2 + \frac{1}{6} 2^2 + \frac{1}{6} 3^2 + \cdots \frac{1}{6} 6^2$$

$$Var[X] = \quad - 3.5^2 = \frac{35}{12}$$

Ex Poission R.V. $\quad P_x(k) = \dfrac{e^{-\lambda}\lambda^k}{k!} \quad k=0,1,2,\ldots$

$$\text{mean} = E(X) = \sum_{k} k\, p_x(x_0)$$

$$= \sum_{k=0}^{\infty} k\, e^{-\lambda} \frac{\lambda^k}{k!}$$

$$= \sum_{k=1}^{\infty} k\, e^{-\lambda} \frac{\lambda^k}{k!}$$

$$= \lambda \sum_{k=1}^{\infty} e^{-\lambda} \frac{\lambda^{k-1}}{(k-1)!} \qquad k-1=m$$

$$= \lambda \underbrace{\sum_{m=0}^{\infty} e^{-\lambda} \frac{\lambda^m}{m!}}_{1}$$

$$\text{mean} = \lambda$$

Var [poisson] = λ

---

joint PMF of multiple R.V.

R.V. x, y coming out of same exp.

joint PMF: $P_{x,y}(x_0, y_0) = P(x = x_0, y = y_0)$

- Roll a die. $X = (\text{roll})^2$ $P_{x,y}$

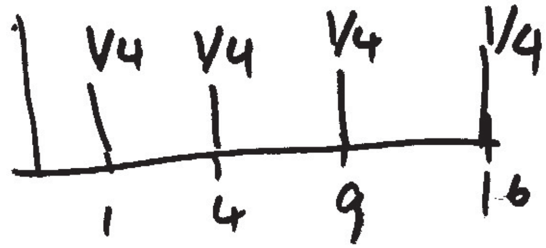
  4 sided. $Y = \text{roll} - 2$

  1, 2, 3, 4

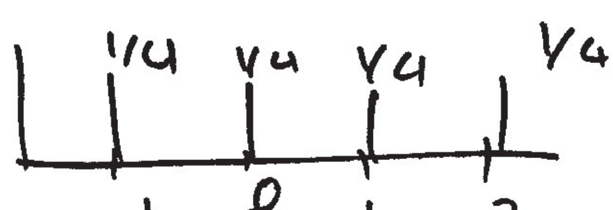
$x = 1, 4, 9, 16$

$y = -1, 0, 1, 2$

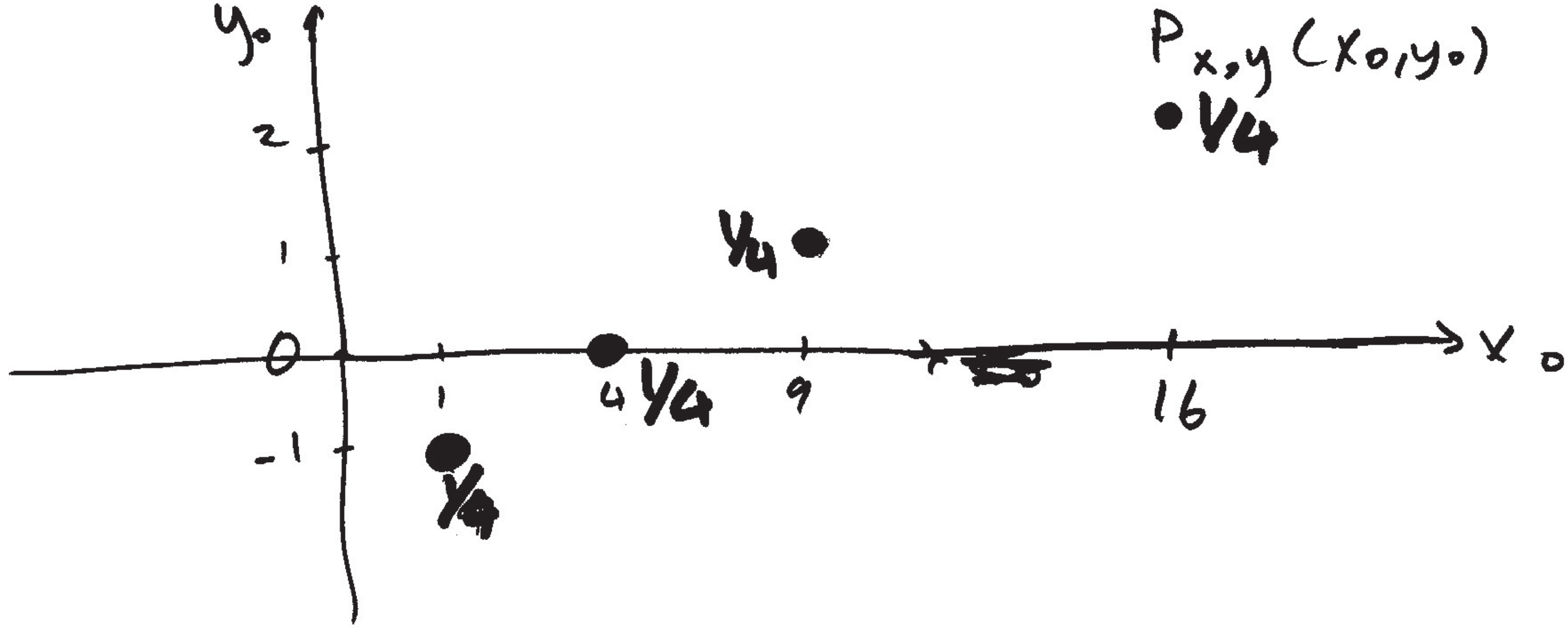
$P_x(x_0)$



$P_y(y_0)$



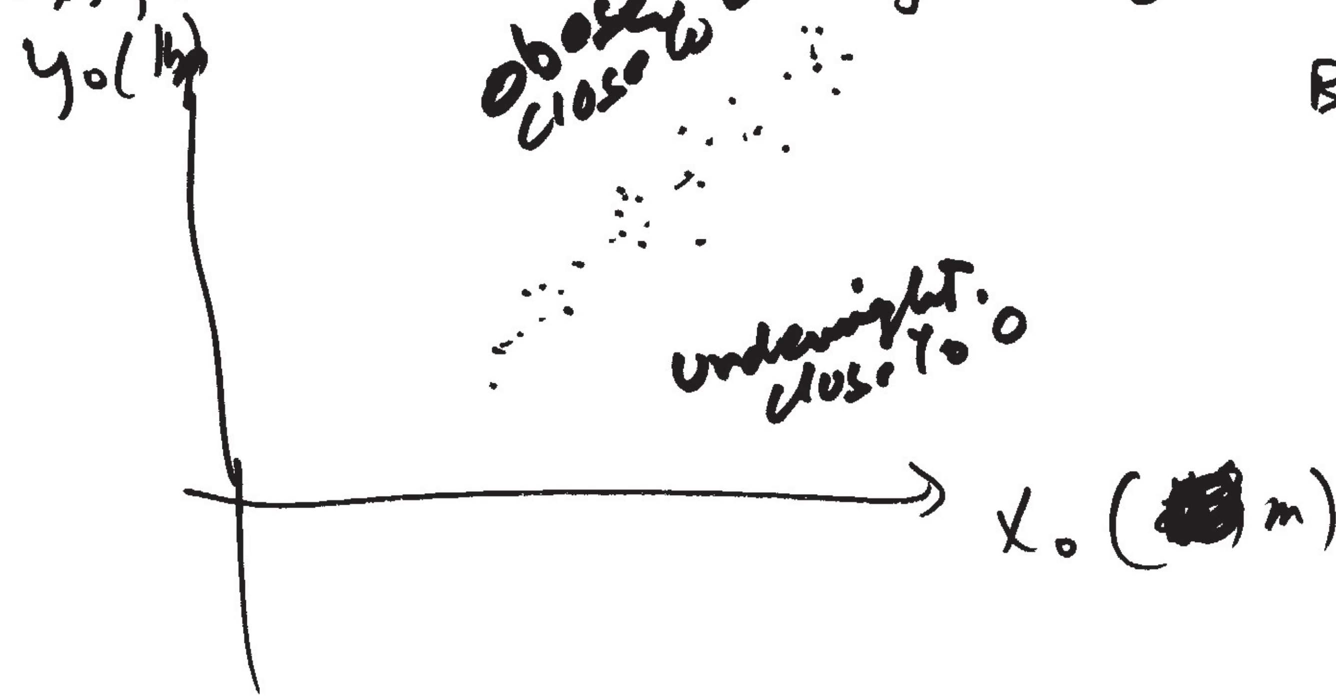
$P_{x,y}(x_0, y_0)$



Ex $P_{x,y}(x_0, y_0)$

X = height
Y = weight

BMI = $\frac{\text{weight in kg}}{(\text{height m})^2}$



$$P_x(x_0) = \sum_{y_0} P_{x,y}(x_0, y_0)$$

$$P_y(y_0) = \sum_{x_0} P_{x,y}(x_0, y_0)$$

joint pmf.

marginal pmf

$P_{x,y}(x_0, y_0)$

y_0 \ x_0	1	2	3	4
4	0	1/20	1/20	1/20
3	1/20	2/20	3/20	1/20
2	1/20	2/20	3/20	1/20
1	1/20	1/20	1/20	0

$p(y=4) = 3/20$

$p(x=1) = 3/20$

Functions of Multiple R.V.

$$z = g(x,y) \qquad g(x,y) = x + 2y = z$$

$$P_z(z_0) = \sum_{\{(x_0,y_0) \mid g(x_0,y_0) = z_0\}} P_{x,y}(x_0,y_0)$$

$$E[g(x,y)] = \sum_{x_0} \sum_{y_0} g(x_0,y_0)\, P_{x,y}(x_0,y_0)$$

$$E[ax + by + c] = a\,E(x) + b\,E(y) + c.$$

① $z = \begin{cases} 3 \longrightarrow 1/20 \\ 4 \longrightarrow 1/20 \\ 5 \longrightarrow 2/20 \\ 6 \longrightarrow \\ 7 \longrightarrow \\ 8 \longrightarrow \\ 9 \text{ ———} \\ 10 \text{ ———} \\ 11 \text{ ———} \\ 12 \text{ ———} \end{cases}$

$z = 5$

$\begin{cases} x_0 = 3 \\ y_0 = 1 \end{cases} \quad \frac{1}{20}$

$\begin{cases} y_0 = 2 \\ x_0 = 1 \end{cases} \quad 1/20$

$$E[z] = \sum_{z_0} z_0 \, P_z(z_0) = 7.55$$

② Alitor

$$\longrightarrow E[g(x,y)] = E(x) + 2E(y) = 7.55$$

Approach 2 is easier.

General:

$$E[aX + bY + cZ + d] =$$

R.V. R.V R.V.

$$= a\,E(X) + b\,E(Y) + c\,E(Z) + d$$

Expectation is a linear operator.

Ex mean of a binomial R.V.

$$X_i = \begin{cases} 1 & \text{student gets A} \\ 0 & \text{otherwise.} \end{cases}$$

Bernoulli R.V.

$X_1, \ldots\ldots X_n$

300 students in class.

$P(\text{getting A}) = 1/3$

X = # of students that get an A.

$= X_1 + X_2 + \cdots \; X_n$

<u>Q</u> what is $E(X)$?

$$E(X) = E[X_1] + E[X_2] + \cdots \; E[X_{300}]$$

$$= \underbrace{\tfrac{1}{3} + \tfrac{1}{3} + \cdots + \tfrac{1}{3}}_{300}$$

$$E(X) = 100$$

<u>Q</u> general Bernoulli trial n times. prob success is p.

$$E[X] = \sum_{i=1}^{n} E[X_i] = \sum_{i=1}^{n} p = pn.$$

<u>English</u> : Expected value of sum $=$ sum of expected values.

Conditioning

Conditional PMF of R.V x conditional upon event A.

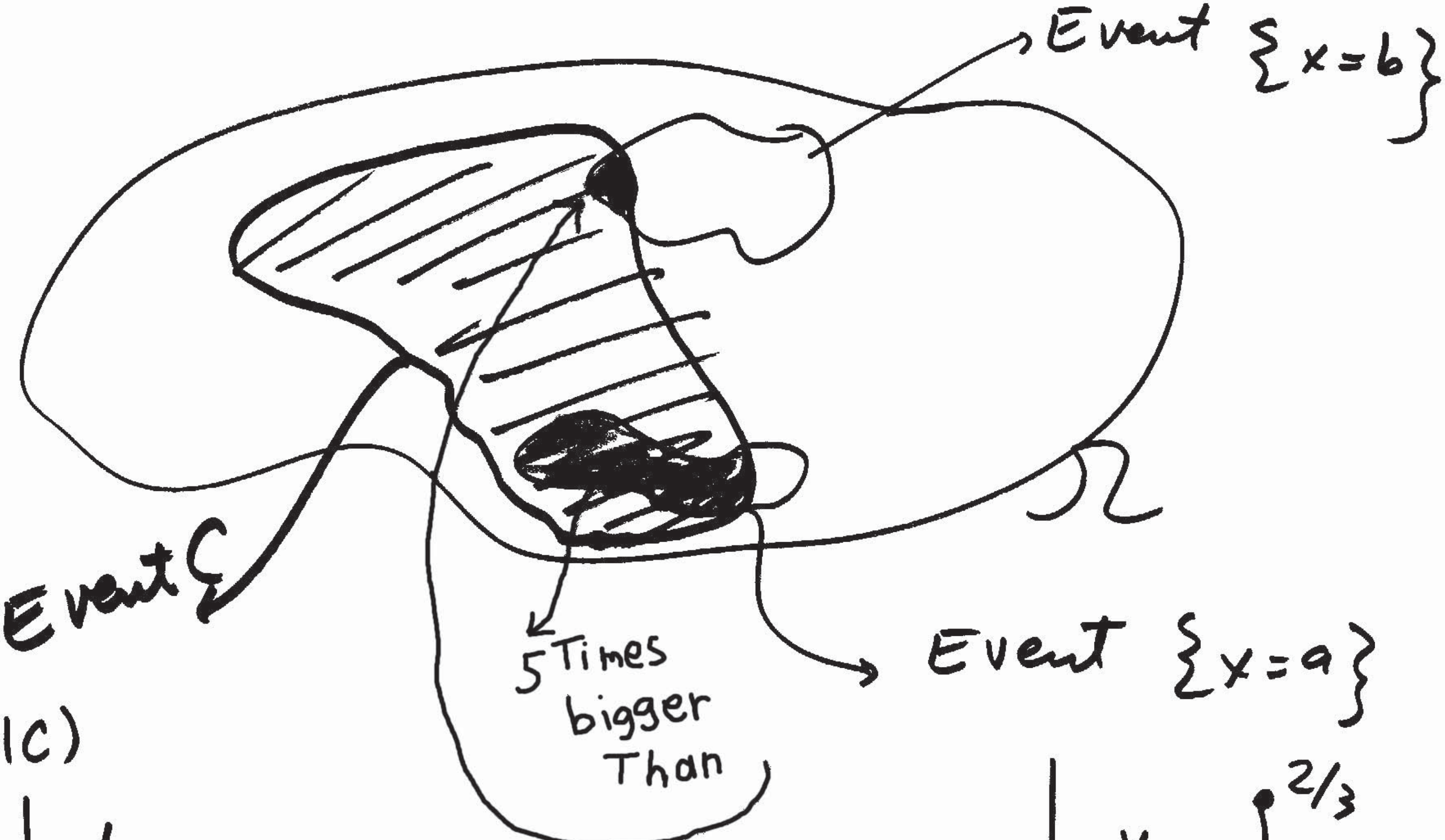
$$P_{X|A}(x_0) = p(X = x_0 / A) = \frac{P(\{X = x_0\} \cap A)}{P(A)}$$

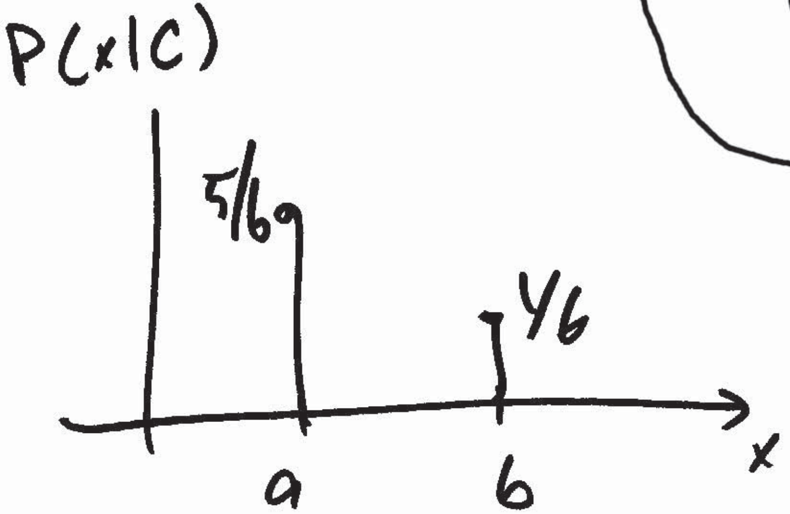
Ex X = roll of a die. 6 sided, unbiased.

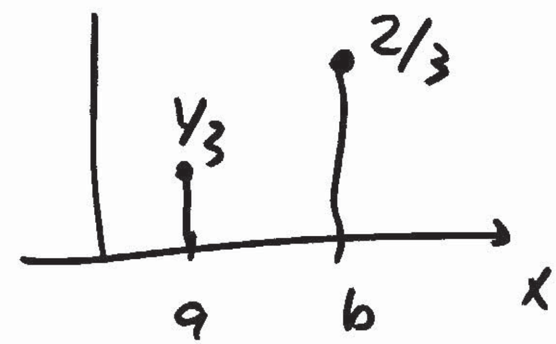
A = event it is even

$$p(x|A) = P(x = k \mid \text{roll is even})$$

$$= \begin{cases} 1/3 = \dfrac{1/6}{1/2} = \dfrac{1}{3} & k = 2, 4, 6 \\ 0 & \text{otherwise} \end{cases}$$

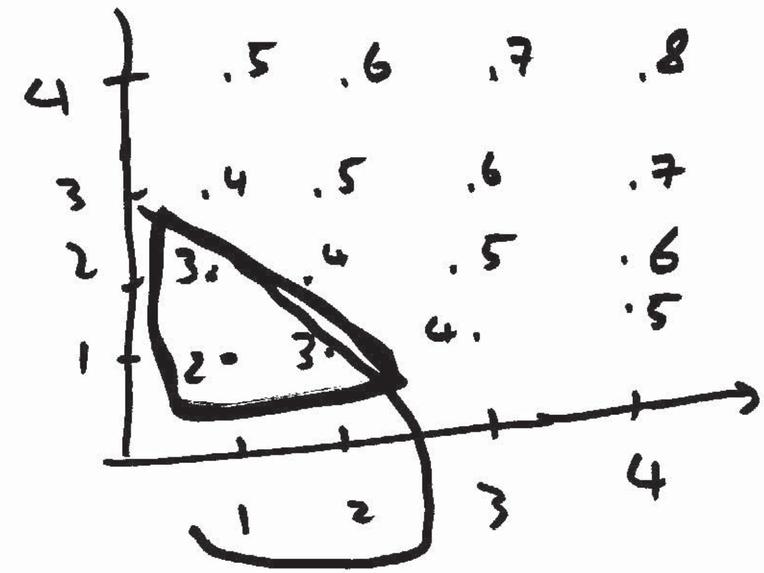

Event {x=b}
Event {
5 Times bigger Than
Event {x=a}


P(x|C)
5/6
1/6
a
b
x


2/3
1/3
a
b
x

Ex roll a die Twice. X = sum.

$$P_X(x_0) = \begin{cases} 1/16 & \to x_0 = 2 \\ 2/16 & \to x_0 = 3 \\ 3/16 & \to 4 \\ 4/16 & \to 5 \\ 3/16 & \to 6 \\ 2/16 & \to 7 \\ 1/16 & \to 8 \end{cases}$$

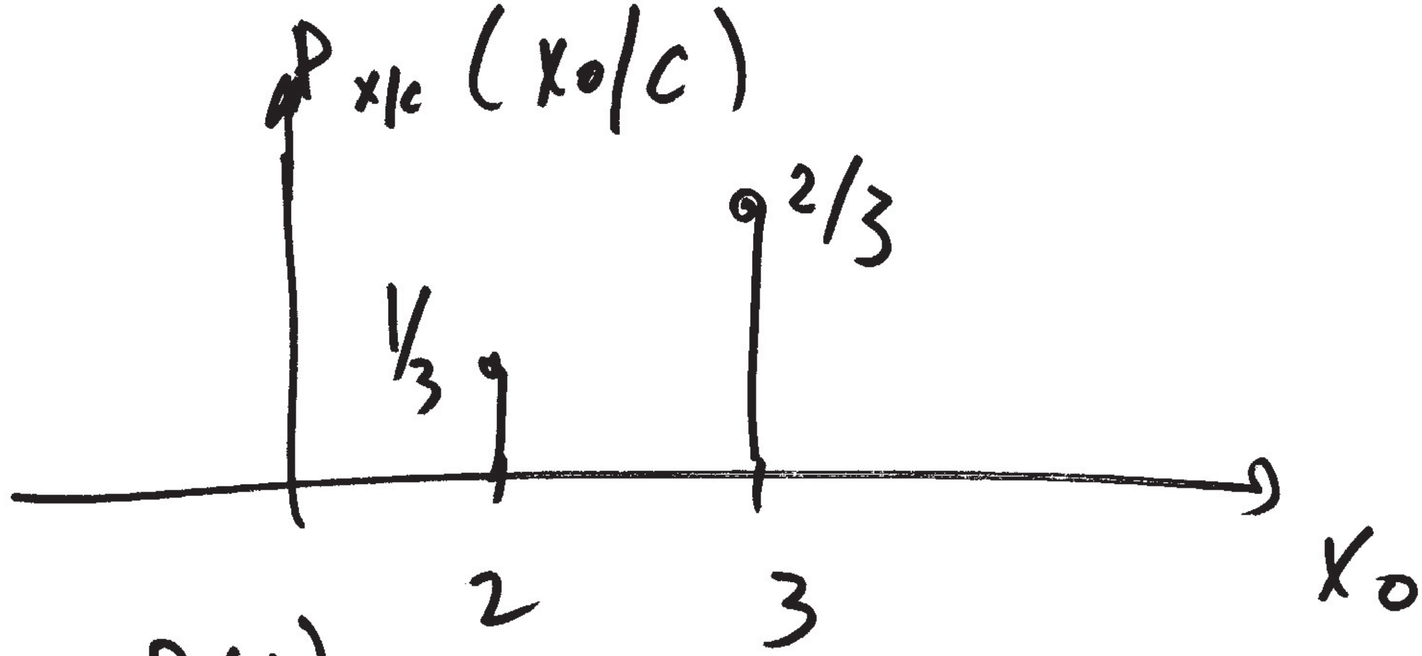


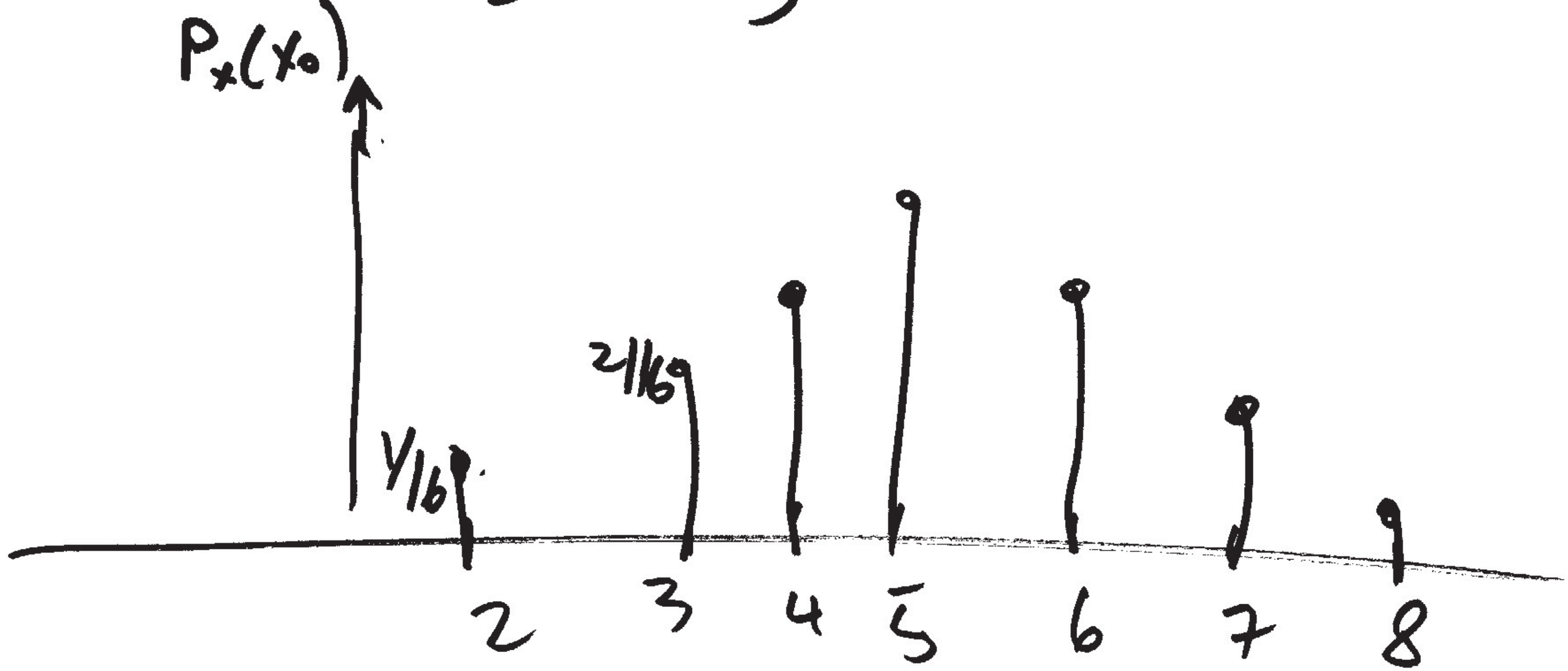
Event C = sum < 4

$$P_{X|C}(x_0 \mid C) = \begin{cases} 1/3 = \frac{P(X=2, C)}{P(C)} & X = 2 \\ 2/3 & X = 3 \\ \emptyset & \text{otherwise} \end{cases}$$

$$\frac{1/16}{3/16} = 1/3$$

$$\frac{P(X=3, C)}{P(C)} = \frac{2/16}{3/16} = 2/3$$


P x|c (x0|c)
2/3
1/3
2
3
x0


Px(x0)
2/16
1/16
2
3
4
5
6
7
8

Conditioning of one R.V. on another R.V.

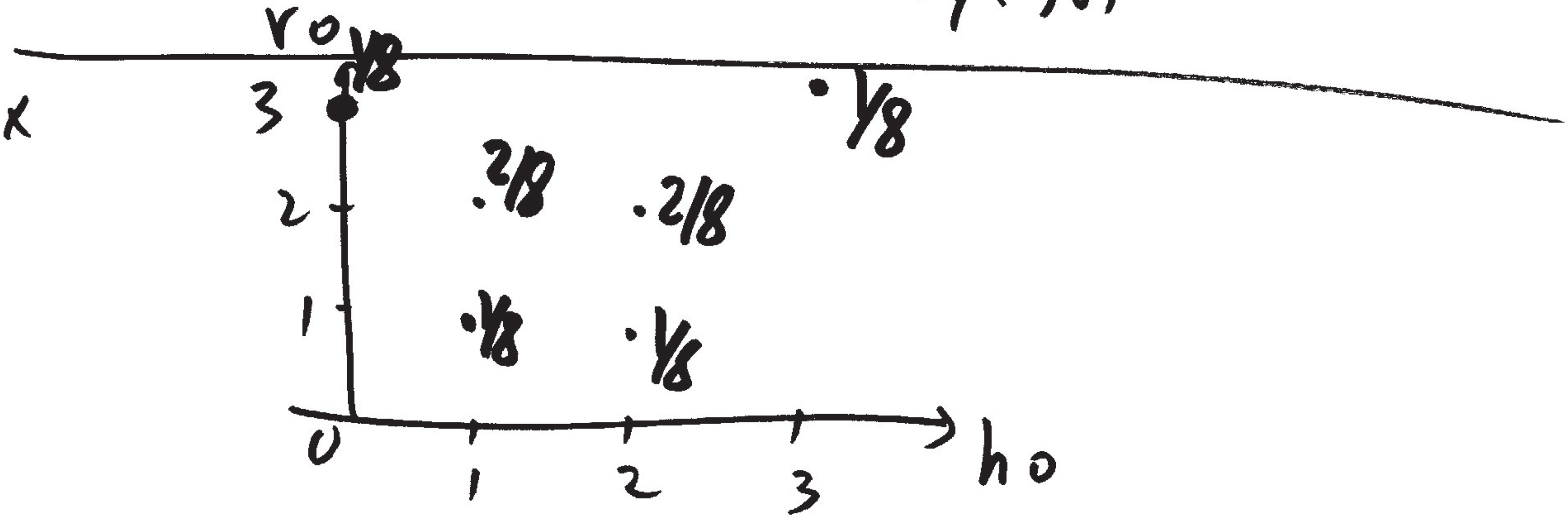
Conditional PMF.

$$P_{X|Y}(x_0/y_0) \triangleq P(X = x_0 \,/\, Y = y_0)$$

$$= \frac{P(X = x_0,\ Y = y_0)}{P(Y = y_0)}$$

$$= \frac{P_{X,Y}(x_0, y_0)}{P_Y(y_0)}$$

Ex



compute $P_{r/h}(r_0/2) = \begin{cases} 1/3 = \dfrac{1/8}{3/8} = 1/3 & r_0 = 1 \\ 2/3 = \dfrac{2/8}{3/8} = 2/3 & r_0 = 2 \\ \phi & \text{otherwise} \end{cases}$

$$P_x(x_0) = \sum_{y_0} P_{x,y}(x_0, y_0)$$

$$P_x(x_0) = \sum_{y_0} \underbrace{P_{x|y}(x_0/y_0)}_{\text{conditional distrib}} P_y(y_0)$$

marginal.