UC Berkeley
Department of Electrical Engineering and Computer Sciences

ELECTRICAL ENGINEERING 126: PROBABILITY AND RANDOM PROCESSES

Problem Set 11
Fall 2017

Issued: November 22, 2017 **Due:** 9 AM, Friday, December 1, 2017

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1. **Continuous-Time Markov Chains: Introduction**

Consider the continuous-time Markov chain with state space $\{1, 2, 3, 4\}$ and the rate matrix

$$Q = \begin{bmatrix} -3 & 1 & 1 & 1 \\ 0 & -3 & 2 & 1 \\ 1 & 2 & -4 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}.$$

(a) Find the stationary distribution π.

(b) Suppose the chain starts in state 1. What is the expected amount of time until it changes state for the first time?

(c) Again assume the chain starts in state 1. What is the expected amount of time until the chain is in state 4?

2. $M/M/2$ **Queue**

A queue has Poisson arrivals with rate λ. It has two servers that work in parallel. Where there are at least two customers in the queue, two are being served. When there is only one customer, only one server is active. The service times are i.i.d. exponential random variables with rate μ.

(a) Argue that the queue length is a Markov chain.

(b) Draw the state transition diagram.

(c) Find the minimum value of μ so that the queue is positive-recurrent (i.e., admits a stationary distribution) and solve the balance equations.

3. **Links**

Consider the graph shown in Figure 1. There are two parallel directed paths from the source node S to the destination node D. One of these is the path comprised of the two successive directed links 1 and 2, going through the intermediate node I. The other is the direct path comprised of the directed link 3. At any time, each of the links is in one of two state: *on* or *off*. Link i switches between its on and off states at rate λ_i, $i = 1, 2, 3$, independently over links. The rates here refer to the rate in exponential distribution. Thus the state of each link can be modeled as a two-state continuous-time Markov chain. We will say that S is connected to D at any time iff there is a path from S to D comprised of on links.

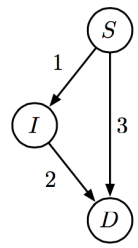

Figure 1: Source to destination links.

(a) What is the stationary probability that S is connected to D?

(b) Assume that the process is in stationarity. Condition on S being connected to D at time 0, and let δ_t, for $t > 0$, denote the conditional probability that S is not connected to D at time t. What is $\frac{\mathrm{d}}{\mathrm{d}t}\delta_t\Big|_{t=0}$ (i.e., the first derivative of δ_t at time 0)?

(c) Assume that the process is in stationarity. Condition on S being not connected to D at time 0. What is the conditional mean time it takes for S to be connected to D?

4. Poisson Queues

A continuous-time queue has Poisson arrivals with rate λ, and it is equipped with infinitely many servers. The servers can work in parallel on multiple customers, but they are non-cooperative in the sense that a single customer can only be served by one server. Thus, when there are k customers in the queue ($k \in \mathbb{N}$), k servers are active. Suppose that the service time of each customer is exponentially distributed with rate μ and they are i.i.d.

(a) Argue that the queue-length is a Markov chain. Draw the transition diagram of the Markov chain.

(b) Prove that for all finite values of λ and μ the Markov chain is positive-recurrent and find the invariant distribution.

5. Taxi Queue

Empty taxis pass by a street corner at a Poisson rate of two per minute and pick up a passenger if one is waiting there. Passengers arrive at the street corner at a Poisson rate of one per minute and wait for a taxi only if there are less than four persons waiting; otherwise they leave and never return. John arrives at the street corner at a given time. Find his expected waiting time, given that he joins the queue. Assume that the process is in steady state.

6. Two-Server System

A company has two servers (the second server is a backup in case the first server fails, so only one server is ever used at a time). When a server is running, the time until it breaks down is exponentially distributed with rate μ. When a server is broken, it is taken to the repair shop. The repair shop can only fix one server at a time, and its repair time is exponentially distributed with rate λ. Find the long-run probability that no servers are operational.