UC Berkeley
Department of Electrical Engineering and Computer Sciences

ELECTRICAL ENGINEERING 126: PROBABILITY AND RANDOM PROCESSES

Problem Set 7
Fall 2017

Issued: October 12, 2017 **Due:** 9 AM, Thursday, October 19, 2017

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1. **Three-State Chain**

   Consider the Markov chain of Figure 1, where $a, b \in (0, 1)$.

   (a) Find the invariant distribution.

   (b) Calculate $\mathbb{P}(X(1) = 1, X(2) = 0, X(3) = 0, X(4) = 1 \mid X(0) = 0)$.

   (c) Show that the Markov chain is aperiodic.

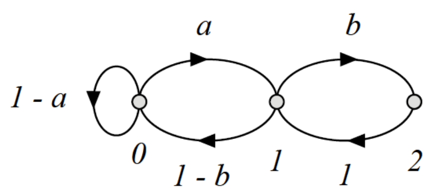


Figure 1: Markov chain for Problem 1.

2. **Finite Random Walk**

   (a) Find the steady-state probabilities $\pi_0, \dots, \pi_{k-1}$ for the Markov chain in Figure 2. Here, k is a positive integer and $p \in (0, 1)$. Express your answer in terms of the ratio $\rho = p/q$, where $q = 1 - p$. Pay particular attention to the special case $\rho = 1$.

   (b) Find the limit of π_0 as k approaches infinity; give separate answers for $\rho < 1$, $\rho = 1$, and $\rho > 1$. Find limiting values of π_{k-1} for the same cases.

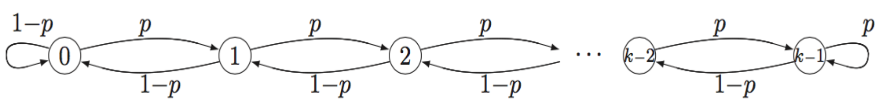


Figure 2: Markov chain for Problem 2.

3. **Fly on a Graph**

   A fly wanders around on a graph G with vertices $V = \{1, \dots, 5\}$, shown in Figure 3.

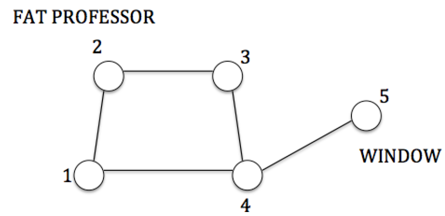


Figure 3: A fly wanders randomly on a graph.

(a) Suppose that the fly wanders as follows: if it is at node i at time n, then it chooses one of its neighbors j of i uniformly at random, and then wanders to node j at time $n+1$. For times $n = 0, 1, 2, \ldots$, let X_n be the fly's position at time n. Argue that $\{X_n,\ n \in \mathbb{N}\}$ is a Markov chain, and find the invariant distribution.

(b) Now for the process in part (a), suppose that the (not-to-be-named) professor sits at node 2 reading a heavy book. The professor is very fat, so he/she doesn't move at all, but will drop the book on the fly if it reaches node 2 (killing it instantly). On the other hand, node 5 is a window that lets the fly escape. What is the probability that the fly escapes through the window supposing that it starts at node 1?

(c) Now suppose that the fly wanders as follows: when it is at node i at time n, it chooses uniformly from all neighbors of node i except for the one that it just came from. For times $n = 0, 1, 2, \ldots$, let Y_n be the fly's position at time n. Is this new process $\{Y_n, n \in \mathbb{N}\}$ a Markov chain? If it is, write down the probability transition matrix; if not, explain why it does not satisfy the definition of Markov chains.

4. **Product of Rolls of a Die**

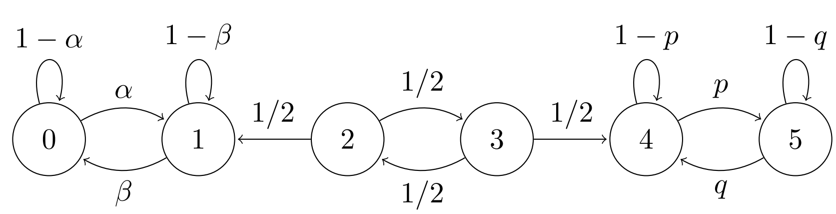
A fair die with labels (1 to 6) is rolled until the product of the last two rolls is 12. What is the expected number of rolls?

5. **Ant**

An ant is walking on the non-negative integers. At each step, the ant moves forward one step with probability $p \in (0, 1)$, or slides back down to 0 with probability $1 - p$. What is the average time it takes for the ant to get to n, where n is a positive integer?

6. **Reducible Markov Chain**

Consider the following Markov chain, for $\alpha, \beta, p, q \in (0, 1)$.



(a) What are all of the communicating classes? (Two nodes x and y are said to belong to the same communicating class if x can reach y and y can reach x through paths of positive probability.) For each communicating class, classify it as recurrent or transient.

(b) Given that we start in state 2, what is the probability that we will reach state 0 before state 5?

(c) What are all of the possible stationary distributions of this chain? (Note that there is more than one.)

(d) Suppose we start in the initial distribution $\pi_0 := \begin{bmatrix} 0 & 0 & \gamma & 1-\gamma & 0 & 0 \end{bmatrix}$ for some $\gamma \in [0, 1]$. Does the distribution of the chain converge, and if so, to what?