1. Estimating an Exponential Distribution

You draw a sample $X_1, \ldots, X_n$ ($n$ is a positive integer) for the lifetime of a light bulb (assumed to be exponentially distributed).

(a) Frankly, you are beginning to think that your boss is unreasonable. She makes the following demands:

- You have exactly 3 samples, with $X_1 = 3$, $X_2 = 7$, $X_3 = 5$. That is, the first light bulb dies after 3 seconds, the second light bulb dies after 7 seconds, and the third light bulb dies after 5 seconds.
- Your confidence interval is specified to be the interval $(0, \varepsilon)$.
- You are no longer estimating the mean lifetime; now you are estimating the rate $\lambda$ at which the light bulbs die.
- Your confidence interval must be exact: you may not use any inequalities or approximations.

What is your reported confidence, in terms of $\varepsilon$? [Please do not leave your answers in terms of an integral.]

(b) In the same setting as the previous part, can you give an exact confidence interval for the mean light bulb lifetime with the same confidence level as before? (*Hint: You don’t need to complete Part (c) to answer this successfully.*)

2. Chernoff Bound Application: Load Balancing

Here, we will give an application for the Chernoff bound which is instrumental for calculating confidence intervals. However, we will need a slightly more general version of the bound that works for any Bernoulli random variables. For any positive integer $n$, if $X_1, \ldots, X_n$ are i.i.d. Bernoulli, with $\Pr(X_i = 1) = p$, and $S_n = \sum_{i=1}^{n} X_i$, then the following bound holds for $0 \leq \varepsilon \leq 1$:

$$\Pr(S_n > (1 + \varepsilon)np) \leq \exp\left(-\frac{\varepsilon^2 np}{3}\right). \quad (1)$$

You may take (1) as a fact (or try to prove it on your own if you want!).

Here is the setting: there are $k$ ($k$ a positive integer) servers and $n$ users. The simplest load balancing scheme is simply to assign each user to a server chosen uniformly at random (think of the users as “balls” and we are tossing them into server “bins”). By using the union bound, show that with probability at least $1 - 1/k^2$, the maximum load of any server is at most $n/k + 3\sqrt{\ln k} \sqrt{n/k}$.
3. **Basic Properties of Jointly Gaussian Random Variables**

Prove that a collection of jointly Gaussian random variables $X_1, \ldots, X_n$ ($n$ is a positive integer) are independent if and only if they are uncorrelated. Also show that any linear combination of these random variables will be a Gaussian random variable. [Hint: For first part, use the characteristic function definition, and look at the covariance matrix for uncorrelated RVs.]

4. **Gaussian Hypothesis Testing**

Consider a hypothesis testing problem that if $X = 0$, you observe a sample of $\mathcal{N}(\mu_0, \sigma^2)$, and if $X = 1$, you observe a sample of $\mathcal{N}(\mu_1, \sigma^2)$, where $\mu_0, \mu_1 \in \mathbb{R}$, $\sigma^2 > 0$. Find the Neyman-Pearson test for false alarm $\alpha \in (0,1)$, that is, $P(\hat{X} = 1 \mid X = 0) \leq \alpha$.

5. **Hypothesis Test for Uniform Distribution**

If $X = 0$, $Y \sim \text{Uniform}[-1, 1]$ and if $X = 1$, $Y \sim \text{Uniform}[0, 2]$. Solve a hypothesis testing problem so that the probability of false alarm is less than or equal $\beta \in (0,1)$.

6. **BSC Hypothesis Testing**

You are testing a digital link that corresponds to a BSC with some error probability $\epsilon \in [0, 0.5)$. You observe $n$ inputs and outputs of the BSC, where $n$ is a positive integer. You want to solve a hypothesis problem to detect that $\epsilon > 0.1$ with a probability of false alarm at most equal to 0.05. Assume that $n$ is very large and use the CLT.