UC Berkeley
Department of Electrical Engineering and Computer Sciences

ELECTRICAL ENGINEERING 126: PROBABILITY AND RANDOM PROCESSES

Midterm 1 Review
Fall 2017

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1. **Compact Arrays**

   Consider an array of n entries, where n is a positive integer. Each entry is chosen uniformly randomly from $\{0, \dots, 9\}$. We want to make the array more compact, by putting all of the non-zero entries together at the front of the array. As an example, suppose we have the array

   $$[6, 4, 0, 0, 5, 3, 0, 5, 1, 3].$$

   After making the array compact, it now looks like

   $$[6, 4, 5, 3, 5, 1, 3, 0, 0, 0].$$

   Let i be a fixed positive integer in $\{1, \dots, n\}$. Suppose that the ith entry of the array is non-zero (for this question, assume that the array is indexed starting from 1). After making the array compact, the ith entry has been moved to index X. Calculate $\mathbb{E}[X]$ and $\operatorname{var} X$.

2. **Graphical Density**

   Figure 1 shows the joint density $f_{X,Y}$ of the random variables X and Y.

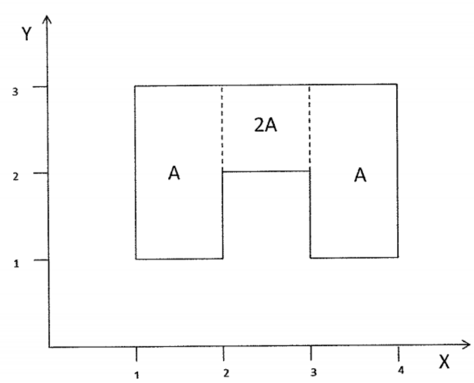

Figure 1: Joint density of X and Y.

   (a) Find A and sketch f_X, f_Y, and $f_{X|X+Y\leq 3}$.

   (b) Find $\mathbb{E}[X \mid Y = y]$ for $1 \leq y \leq 3$ and $\mathbb{E}[Y \mid X = x]$ for $1 \leq x \leq 4$.

   (c) Find $\operatorname{cov}(X, Y)$.

3. **Office Hours**

In an EE 126 office hour, students bring either a difficult-to-answer question with probability $p = 0.2$ or an easy-to-answer question with probability $1-p = 0.8$. A GSI takes a random amount of time to answer a question, with this time duration being exponentially distributed with rate $\mu_D = 1$ (questions per minute)—where D denotes "difficult"—if the problem is difficult, and $\mu_E = 2$ (questions per minute)—where E denotes "easy"—if the problem is easy.

(a) You visit office hours and find a GSI answering the question of another student. Conditioned on the fact that the GSI has been busy with the other students question for $T > 0$ minutes, let q be the conditional probability that the problem is difficult. Find the value of q.

(b) Conditioned on the information above, find the expected amount of time you have to wait from the time you arrive until the other students question is answered.

(c) Now suppose two GSIs share a room and the professor is holding office hours in a different room. Both GSIs in the shared room are busy helping a student, and each has been answering questions for $T > 0$ minutes (there are no other students in the room). The amount of time the professor takes to answer a question is exponentially distributed with rate $\lambda = 6$ regardless of the difficulty. Supposing that the professor's room has two students (one of whom is being helped), in which room should you ask your question?

4. **Exponential Fun**

(a) Let X_1 and X_2 be i.i.d. exponential random variables with parameter λ. Compute the density of $X_1 + X_2$.

(b) Now, for a positive integer n, let $X_1, \dots, X_n$ be i.i.d. exponential random variables with parameter λ and $S := \sum_{i=1}^{n} X_i$. The density of S is given by the n-fold convolution of the exponential distribution with itself. Compute this density.

(c) Using the above result, consider now the random sum $X_1 + \cdots + X_N$, where N is a geometric random variable with parameter p. Compute the density of $X_1 + \cdots + X_N$.

5. **Galton-Watson Branching Process**

Consider a population of N individuals for some positive integer N. Let ξ be a random variable taking values in $\mathbb{N}$ with $\mathbb{E}[\xi] = \mu$ and $\operatorname{var} \xi = \sigma^2$. At the end of each year, each individual, independently of all other individuals and generations, leaves behind a number of offspring which has the same distribution as ξ. For each $n \in \mathbb{N}$, let X_n denote the size of the population at the end of the nth year. Compute $\mathbb{E}[X_n]$ and $\operatorname{var} X_n$. [*Hint*: For the variance, you will need to consider the case when $\mu = 1$ separately from the case when $\mu \neq 1$.]

6. Combining Transforms

Let X, Y, and Z be independent random variables. X is Bernoulli with $p = 1/4$. Y is exponential with parameter 3. Z is Poisson with parameter 5.

(a) Find the transform of $5Z + 1$.

(b) Find the transform of $X + Y$.

(c) Consider the new random variable $U = XY + (1 - X)Z$. Find the transform associated with U.