Problem 1. Statistical Estimation
Given $X \in \{0, 1\}$, the random variable $Y$ is exponentially distributed with rate $3X + 1$.

(a) Assume $\Pr(X = 1) = p \in (0, 1)$ and $\Pr(X = 0) = 1 - p$. Find the MAP estimate of $X$ given $Y$.

(b) Find the MLE of $X$ given $Y$.

Problem 2. Exponential: MLE & MAP
The random variable $X$ is exponentially distributed with mean 1. Given $X$, the random variable $Y$ is exponentially distributed with rate $X$.

1. Find $MLE[X \mid Y]$.

2. Find $MAP[X \mid Y]$.

Problem 3. Laplace Prior & $\ell^1$-Regularization
Suppose you draw $n$ i.i.d. data points $(x_1, y_1), \ldots, (x_n, y_n)$, where $n$ is a positive integer and the true relationship is $Y = WX + \varepsilon$, $\varepsilon \sim \mathcal{N}(0, \sigma^2)$. (That is, $Y$ has a linear dependence on $X$, with additive Gaussian noise.) Further suppose that $W$ has a prior distribution with density

$$f_W(w) = \frac{1}{2\beta} \exp\left(-\frac{|w|}{\beta}\right), \quad \beta > 0.$$  

(This is known as the Laplace distribution.) Show that finding the MAP estimate of $W$ given the data points $\{(x_i, y_i) : i = 1, \ldots, n\}$ is equivalent to minimizing the cost function

$$J(w) = \sum_{i=1}^n (y_i - wx_i)^2 + \lambda|w|$$

(you should determine what $\lambda$ is). This is interpreted as a one-dimensional $\ell^1$-regularized least-squares criterion, also known as LASSO.