**Problem 1. Breaking a Stick**  
I break a stick $n$ times, where $n$ is a positive integer, in the following manner: the $i$th time I break the stick, I keep a fraction $X_i$ of the remaining stick where $X_i$ is uniform on the interval $[0, 1]$ and $X_1, X_2, \ldots, X_n$ are i.i.d. Let $P_n = \prod_{i=1}^{n} X_i$ be the fraction of the original stick that I end up with.

1. Show that $P_n^{1/n}$ converges almost surely to some constant function.

2. Compute $E[P_n]^{1/n}$.

**Solution 1.**

1. 
\[
\lim_{n \to \infty} \ln P_n^{1/n} = \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \ln X_i = E[\ln X_1], \text{ a.s.,}
\]

where the second equality follows from the Strong Law of Large Numbers. Note that $E[\ln X_1] = \int_{0}^{1} \ln x \, dx = -1$. Thus, from the continuous mapping theorem we have that:
\[
\lim_{n \to \infty} P_n^{1/n} = \lim_{n \to \infty} \exp \ln P_n^{1/n} = \frac{1}{e}, \text{ a.s.}
\]

2. We note that
\[
E[P_n] = E\left[\prod_{i=1}^{n} X_i\right] = E[X_1^n] = \left(\frac{1}{2}\right)^n,
\]

hence
\[
E[P_n]^{1/n} = \frac{1}{2}.
\]

**Problem 2. Mean Square Convergence**

A sequence of random variables $\{X_n\}_{n \geq 0}$, each satisfying $E[X_n^2] < \infty$, is said to converge in mean square to a random variable $X$ if
\[
\lim_{n \to \infty} E[(X_n - X)^2] = 0.
\]

1. Show that convergence in mean square implies convergence in probability.

2. Consider the sequence of random variables $(X_n)_{n \geq 1}$, where each $X_n \sim \text{Bernoulli}(1/n)$. Show that this sequence converges to 0 in mean square.
3. Does it converge almost surely?

**Solution 2.**
1. Assume that \( \mathbb{E}[(X_n - X)^2] \to 0 \), as \( n \to \infty \). For any \( \epsilon > 0 \),
   \[
   \Pr(|X_n - X| > \epsilon) = \Pr((X_n - X)^2 > \epsilon^2) \leq \frac{\mathbb{E}((X_n - X)^2)}{\epsilon^2} \to 0, \quad \text{as } n \to \infty.
   \]
2. \( \mathbb{E}[X_n^2] = \mathbb{E}[X_n] = \frac{1}{n} \to 0, \quad \text{as } n \to \infty. \)
3. It does not converge almost surely though, since for any \( \epsilon \in (0, 1) \) and any \( m \)
we have that
   \[
   \Pr(|X_n - 0| < \epsilon, \text{ for all } n \geq m) = \lim_{n \to \infty} \prod_{i=m}^{n} \left(1 - \frac{1}{i}\right) = \lim_{n \to \infty} \prod_{i=m}^{n} \left(\frac{i-1}{i}\right)
   = \lim_{n \to \infty} \frac{m-1}{m} \cdot \frac{m}{m+1} \cdots \frac{n-1}{n} = \lim_{n \to \infty} \frac{m-1}{n} = 0.
   \]
Now take the union over \( m \).
   \[
   \Pr(\exists m \text{ such that } |X_n| < \epsilon \text{ for all } n \geq m)
   = \Pr\left(\bigcup_{m=1}^{\infty} \{|X_n| < \epsilon \text{ for all } n \geq m\}\right)
   \leq \sum_{m=1}^{\infty} \Pr(|X_n| < \epsilon \text{ for all } n \geq m) = 0.
   \]
This implies that \( \Pr(\lim_{n \to \infty} X_n = 0) = 0 \), which means \( (X_n)_{n=1}^{\infty} \) does not converge a.s. to 0. Since (a) and (b) imply that \( X_n \to 0 \) in probability as \( n \to \infty \), if \( (X_n)_{n=1}^{\infty} \) were to converge to a random variable \( X \) a.s., then \( X \) would have to be 0 (because a.s. convergence implies convergence in probability), but we have seen that \( (X_n)_{n=1}^{\infty} \) does not converge to 0 a.s., which means \( (X_n)_{n=1}^{\infty} \) does not converge a.s. to anything.

**Problem 3. Convergence in Probability**
Let \( (X_n)_{n=1}^{\infty} \), be a sequence of i.i.d. random variables distributed uniformly in \([-1, 1]\).
Show that the following sequences \( (Y_n)_{n=1}^{\infty} \) converge in probability to some limit.

(a) \( Y_n = (X_n)^n \).
(b) \( Y_n = \prod_{i=1}^{n} X_i \).
(c) \( Y_n = \max\{X_1, X_2, \ldots, X_n\} \).
(d) \( Y_n = (X_1^2 + \cdots + X_n^2)/n \).

**Solution 3.** (a) For any \( \epsilon > 0 \),
   \[
   \Pr(|Y_n| > \epsilon) = \Pr(|X_n| > \epsilon^{1/n}) = 1 - \epsilon^{1/n} \to 0 \quad \text{as } n \to \infty.
   \]
Thus, the sequence converges to 0 in probability.
(b) By independence of the random variables,
\[ E[Y_n] = E[X_1] \cdots E[X_n] = 0, \]
\[ \text{Var} Y_n = E[Y_n^2] = (\text{Var} X_1)^n = \left(\frac{1}{3}\right)^n. \]

Now since \( \text{Var} Y_n \to 0 \) as \( n \to \infty \), by Chebyshev’s Inequality the sequence converges to its mean, that is, 0, in probability.

(c) Consider \( \epsilon \in [0, 1] \). We see that:
\[ \Pr(|Y_n - 1| \geq \epsilon) = \Pr(\max\{X_1, \ldots, X_n\} \leq 1 - \epsilon) \]
\[ = \Pr(X_1 \leq 1 - \epsilon, \ldots, X_n \leq 1 - \epsilon) \]
\[ = \Pr(X_1 \leq 1 - \epsilon)^n = \left(1 - \frac{\epsilon}{2}\right)^n. \]

Thus, \( \Pr(|Y_n - 1| \geq \epsilon) \to 0 \) as \( n \to \infty \) and we are done.

(d) The expectation is
\[ E[Y_n] = \frac{1}{n} \cdot n E[X_1^2] = \frac{1}{3}. \]

Then, we bound the variance.
\[ \text{Var} Y_n = \frac{1}{n} \text{Var} X_1^2 \leq \frac{1}{n} \to 0 \quad \text{as} \quad n \to \infty, \]
since \( X_1^2 \leq 1 \). Hence, we see that \( Y_n \to 1/3 \) in probability as \( n \to \infty \).

**Remark:** We now provide an interpretation for the previous result. The sample space for \( Y_n \) is \( \Omega_n = [-1, 1]^n \), which is an \( n \)-dimensional cube. The result we have just proved shows that, for any \( \epsilon > 0 \), the set
\[ B_n = \left\{ x \in \mathbb{R}^n : \frac{1}{3}(1 - \epsilon) \leq \frac{x_1^2 + \cdots + x_n^2}{n} \leq \frac{1}{3}(1 + \epsilon) \right\} \]
makes up “most” of the volume of \( \Omega_n \), in the sense that
\[ \frac{\text{volume}(B_n \cap [-1, 1]^n)}{2^n} \to 1 \quad \text{as} \quad n \to \infty. \]

Since \( B_n \) is close to the boundary of a ball of radius \( \sqrt{n/3} \), the result can be stated facetiously as “nearly all of the volume of a high-dimensional cube is contained in the boundary of a ball”. Although this may seem like a meaningless comment, in fact various phenomena such as these contribute to the so-called “curse of dimensionality” in machine learning, which concerns the sparsity of data in high-dimensional statistics.