**Problem 1. Capacity of Symmetric Channels**

Consider a communication channel with input $X$ and output $Y$, both on the alphabet \{1, 2, 3\}, with $P_{Y|X}$ given as

$$
P_{Y|X}(y|x) = \begin{bmatrix}
0.3 & 0.2 & 0.5 \\
0.5 & 0.3 & 0.2 \\
0.2 & 0.5 & 0.3
\end{bmatrix}
$$

Here, rows correspond to $X$ and columns correspond to $Y$. Therefore, for instance, $P_{Y|X}(2|1) = 0.2$ and $P_{Y|X}(1|2) = 0.5$. Such a channel is called “symmetric”, because all rows of the probability transition matrix are permutations of each other, and so are the columns.

1. Show that irrespective of the input distribution $P_X$, $H(Y|X)$ is always constant. Call this constant $c$ and find it.

2. Show that $I(X;Y) \leq \log 3 - c$, with the constant $c$ found in the previous part.

3. Find a input distribution $P_X(x)$ for which $I(X;Y) = \log 3 - c$.

**Problem 2. Conditional entropy of a random walk**

Consider a Markov chain which is random walk on the following graph

Here at time $k$, if we are at node $1 \leq v \leq 4$, we chose $X_{k+1}$ to be one of the neighbors of $v$ uniformly at random.

1. Show that the chain has a unique stationary distribution $\pi$ and find it.

2. Assume that we initialize the chain at the stationary distribution, i.e. $X_0 \sim \pi$. Find $I(X_0; X_1)$.
Problem 3. Generating Erdős-Renyi Random Graphs

True/False: Let $G_1$ and $G_2$ be independent Erdős-Renyi random graphs on $n$ vertices with probabilities $p_1$ and $p_2$, respectively. Let $G = G_1 \cup G_2$, that is, $G$ is generated by combining the edges from $G_1$ and $G_2$. Then, $G$ is an Erdős-Renyi random graph on $n$ vertices with probability $p_1 + p_2$.

Problem 4. Random Graph

Consider a random undirected graph on $n$ vertices, where each of the $\binom{n}{2}$ possible edges is present with probability $p$ independently of all the other edges. If $p = 0$ we have a fully empty graph with $n$ completely disconnected vertices; in contrast, if $p = 1$, every edge exists, the network is an $n$-clique, and every vertex is a distance one from every other vertex.

1. Fix a particular vertex of the graph, and let $D$ be a random variable which is equal to the degree of this vertex. What is the PMF of $D$? Calculate $\lambda \triangleq \mathbb{E}[D]$.

2. Assume that $p = c/n$ where $c > 0$ is a constant, independent of $n$. For large values of $n$, how you would approximate the PMF of $D$?