Problem 1. Poisson Bounds
Let \( X \) be the sum of 20 i.i.d. Poisson random variables \( X_1, \ldots, X_{20} \) with \( \mathbb{E}[X_1] = 1 \). Use Markov’s Inequality and Chebyshev’s Inequality to find an upper bound of \( \Pr(X \geq 26) \).

Solution 1. 1. Using Markov’s Inequality:
\[
\Pr(X \geq a) \leq \frac{\mathbb{E}[X]}{a},
\]
for all \( a > 0 \). So,
\[
\Pr(X \geq 26) \leq \frac{20}{26} \approx 0.769.
\]

2. Using Chebyshev’s Inequality:
\[
\Pr(|X - \mathbb{E}[X]| \geq c) \leq \frac{\sigma_X^2}{c^2},
\]
so we have
\[
\Pr(|X - 20| \geq 6) \leq \frac{20}{36} \approx 0.5556.
\]

Problem 2. Convergence of Exponentials
Let \( X_1, X_2, \ldots \) be i.i.d. \( \text{Exp}(\lambda) \) random variables. Show that
\[
\Pr\left(\frac{X_n}{\ln n} \geq \varepsilon\right) \rightarrow 0 \quad \text{as } n \rightarrow \infty.
\]

Solution 2. Fix \( \varepsilon > 0 \).
\[
\Pr\left(\frac{X_n}{\ln n} \geq \varepsilon\right) = \Pr(X_n \geq \varepsilon \ln n) = \exp(-\lambda \varepsilon \ln n) = n^{-\lambda \varepsilon} \rightarrow 0
\]
as \( n \rightarrow \infty \).

Problem 3. Transform Practice
Consider a random variable \( Z \) with transform
\[
M_Z(s) = \frac{a - 3s}{s^2 - 6s + 8}, \quad \text{for } |s| < 2.
\]
Calculate the following quantities:
1. The numerical value of the parameter $a$.

2. $\mathbb{E}[Z]$.

3. $\text{Var}(Z)$.

**Solution 3.** 1. By definition, we know that $M_Z(s) = \mathbb{E}[e^{sZ}]$. Thus, we know the following must be true:

$$M_Z(0) = \mathbb{E}[e^{0Z}] = 1 = \frac{a}{8}$$

It follows that $a = 8$.

2.

$$\mathbb{E}[Z] = \frac{d}{ds} M_Z(s) \bigg|_{s=0} = \frac{2}{(4-s)^2} + \frac{1}{(2-s)^2} \bigg|_{s=0} = \frac{3}{8}.$$

3. Note that

$$\mathbb{E}[Z^2] = \frac{d^2}{ds^2} M_Z(s) \bigg|_{s=0} = \frac{4}{(4-s)^3} + \frac{2}{(2-s)^3} \bigg|_{s=0} = \frac{5}{16}.$$

Thus,

$$\text{Var}(Z) = \frac{11}{64}.$$