Problem 1. **Backwards Markov Property**

Let \((X_n)_{n \in \mathbb{N}}\) be a Markov chain with state space \(S\). Show that for every \(m, k \in \mathbb{N}\), with \(m \geq 1\), we have

\[
\Pr(X_k = i_0 \mid X_{k+1} = i_1, \ldots, X_{k+m} = i_m) = \Pr(X_k = i_0 \mid X_{k+1} = i_1),
\]

for all states \(i_0, i_1, \ldots, i_m \in S\).

**Solution 1.** By definition of conditional probability we can write

\[
\Pr(X_k = i_0 \mid X_{k+1} = i_1, \ldots, X_{k+m} = i_m) = \frac{\Pr(X_k = i_0, X_{k+1} = i_1, \ldots, X_{k+m} = i_m)}{\Pr(X_{k+1} = i_1, \ldots, X_{k+m} = i_m)}.
\]

Now using the Markov property the numerator can be written as

\[
\Pr(X_k = i_0, X_{k+1} = i_1) \prod_{j=2}^{m} \Pr(X_{k+j} = i_j \mid X_{k+j-1} = i_{j-1}),
\]

and the denominator can be written as

\[
\Pr(X_{k+1} = i_1) \prod_{j=2}^{m} \Pr(X_{k+j} = i_j \mid X_{k+j-1} = i_{j-1}).
\]

So the products cancel out and we obtain

\[
\Pr(X_k = i_0 \mid X_{k+1} = i_1, \ldots, X_{k+m} = i_m) = \frac{\Pr(X_k = i_0, X_{k+1} = i_1)}{\Pr(X_{k+1} = i_1)} = \Pr(X_k = i_0 \mid X_{k+1} = i_1).
\]

Problem 2. **Reducible Markov Chain**

Consider the following Markov chain, for \(\alpha, \beta, p, q \in (0, 1)\).
1. What are all of the communicating classes? (Two nodes $x$ and $y$ are said to belong to the same communicating class if $x$ can reach $y$ and $y$ can reach $x$ through paths of positive probability.) For each communicating class, classify it as recurrent or transient.

2. Given that we start in state 2, what is the probability that we will reach state 0 before state 5?

3. What are all of the possible stationary distributions of this chain? (Note that there is more than one.)

4. Suppose we start in the initial distribution $\pi_0 := \begin{bmatrix} 0 & 0 & \gamma & 1 - \gamma & 0 & 0 \end{bmatrix}$ for some $\gamma \in [0, 1]$. Does the distribution of the chain converge, and if so, to what?

**Solution 2.**

1. The communicating classes are $\{0, 1\}$ (recurrent), $\{4, 5\}$ (recurrent), and $\{2, 3\}$ (transient).

2. Let $T_0$ and $T_5$ denote the time it takes to reach states 0 and 5 respectively. (Note that exactly one of $T_0$ and $T_5$ will be finite.) We are looking to compute $Pr_2(T_0 < T_5)$, and we can set up hitting equations:

   \[
   Pr_2(T_0 < T_5) = \frac{1}{2} + \frac{1}{2} Pr_3(T_0 < T_5),
   \]

   \[
   Pr_3(T_0 < T_5) = \frac{1}{2} Pr_2(T_0 < T_5).
   \]

   Thus, $Pr_2(T_0 < T_5) = \frac{2}{3}$.

3. First we observe that no stationary distribution can put positive probability on a transient state, so the stationary distribution is supported on the states $\{0, 1, 4, 5\}$. Next, if we restrict our attention to only the states $\{0, 1\}$, then we have an irreducible Markov chain with stationary distribution

   \[
   \pi_1 := \frac{1}{\alpha + \beta} \begin{bmatrix} \beta & \alpha \end{bmatrix},
   \]

   and similarly, if we restrict our attention to only the states $\{4, 5\}$, then again we have an irreducible Markov chain with stationary distribution

   \[
   \pi_2 := \frac{1}{p + q} \begin{bmatrix} q & p \end{bmatrix}.
   \]

   Any stationary distribution for the entire chain must be some convex combination of these two stationary distributions. Explicitly, the stationary distributions are of the form

   \[
   \pi = \begin{bmatrix} \frac{c\beta}{\alpha + \beta} & \frac{ca}{\alpha + \beta} & 0 & 0 & (1 - c)q & (1 - c)p \\ \frac{\alpha}{\alpha + \beta} & \frac{\beta}{\alpha + \beta} & p + q & p + q \end{bmatrix}
   \]

   for some $c \in [0, 1]$. 


4. Indeed the distribution will converge, even though we do not have irreducibility. The intuition is as follows. The probability will leak out of the transient states \{2, 3\} until all of the probability mass is supported on the recurrent states. The two recurrent classes can each be considered to be an irreducible aperiodic Markov chain and so the probability mass which enters a recurrent class will settle into equilibrium. To aid us in finding the limiting distribution, we can use the results of Part (b). With probability \(\gamma\), we start in state 2, and with a further probability \(\frac{2}{3}\) we end up in the recurrent class \{0, 1\}. By symmetry, the probability that we end up in \{0, 1\} starting form state 3 is \(\frac{1}{3}\). Thus, the total probability mass which settles into the recurrent class \{0, 1\} is \(2\gamma/3 + (1 - \gamma)/3 = 1/3 + \gamma/3\). Then, the probability mass settling in the recurrent class \{4, 5\} is \(2/3 - \gamma/3\). Therefore, the chain converges to the stationary distribution in (1) with \(c = 1/3 + \gamma/3\).

**Problem 3. Fly on a Graph**

A fly wanders around on a graph \(G\) with vertices \(V = \{1, \ldots, 5\}\), shown in Figure 1.

Figure 1: A fly wanders randomly on a graph.

(a) Suppose that the fly wanders as follows: if it is at node \(i\) at time \(n\), then it chooses one of its neighbors \(j\) of \(i\) uniformly at random, and then wanders to node \(j\) at time \(n + 1\). For times \(n = 0, 1, 2, \ldots\), let \(X_n\) be the fly’s position at time \(n\). Argue that \(\{X_n,\ n \in \mathbb{N}\}\) is a Markov chain, and find the invariant distribution.

(b) Now for the process in part (a), suppose that the (not-to-be-named) professor sits at node 2 reading a heavy book. The professor is very fat, so he/she doesn’t move at all, but will drop the book on the fly if it reaches node 2 (killing it instantly). On the other hand, node 5 is a window that lets the fly escape. What is the probability that the fly escapes through the window supposing that it starts at node 1?

(c) Now suppose that the fly wanders as follows: when it is at node \(i\) at time \(n\), it chooses uniformly from all neighbors of node \(i\) except for the one that it just came from. For times \(n = 0, 1, 2, \ldots\), let \(Y_n\) be the fly’s position at time \(n\). Is this new process \(\{Y_n, n \in \mathbb{N}\}\) a Markov chain? If it is, write down the probability transition matrix; if not, explain why it does not satisfy the definition of Markov chains.

**Solution 3.** (a) Given the position of the fly at time \(n\), the distribution of the position of the fly at time \(n + 1\) is conditionally independent of the previous
You

Stairs

Figure 2: Part (a)

positions of the fly before \( n \). Therefore, \( \{X_n, n \in \mathbb{N}\} \) is a Markov chain. We can get the probability transition matrix

\[
P = \begin{bmatrix}
0 & 1/2 & 0 & 1/2 & 0 \\
1/2 & 0 & 1/2 & 0 & 0 \\
0 & 1/2 & 0 & 1/2 & 0 \\
1/3 & 0 & 1/3 & 0 & 1/3 \\
0 & 0 & 0 & 1 & 0
\end{bmatrix}.
\]

According to \( \pi P = \pi \), we get the invariant distribution

\[
\pi = [0.2 \ 0.2 \ 0.2 \ 0.3 \ 0.1].
\]

(b) Let \( p \) be the probability that the fly escapes through the window supposing that it starts at node 1. According to symmetry, starting from node 3, the probability that the fly escapes through the window is also \( p \). Let \( q \) be the probability that the fly escapes through the window supposing that it starts at node 4. We have

\[
p = \frac{1}{2}(0 + q),
\]

\[
q = \frac{1}{3}(1 + p + p).
\]

Then we get \( p = 1/4 \).

(c) No, it is not a Markov chain. According to the definition of the process

\[
\text{Pr}(Y_{n+1} = 1 \mid Y_n = 4, Y_{n-1} = 1) = 0,
\]

while

\[
\text{Pr}(Y_{n+1} = 1 \mid Y_n = 4, Y_{n-1} = 3) = 0.5.
\]

Therefore, given \( Y_n, Y_{n+1} \) and \( Y_{n-1} \) are not conditionally independent. Then the process \( \{Y_n, n \in \mathbb{N}\} \) is not a Markov chain.

**Problem 4. Twitch Plays Pokemon**

After attending an EECS 126 lecture, you went back home and started playing Twitch Plays Pokemon. Suddenly, you realized that you may be able to analyze Twitch Plays Pokemon.

1. The player in the top left corner performs a random walk on the 8 checkered squares and the square containing the stairs. At every step the player is equally likely to move to any of the squares in the four cardinal directions (North, West, East, South) if there is a square in that direction. Find the expected number of moves until the player reaches the stairs in Figure 4.
2. The player randomly walks in the same way as in the previous part. Find the probability that the player reaches the stairs in the bottom right corner in Figure 1.

Solution 4. 1. Using symmetry, the 9 states can be grouped as follows.

\[
\begin{array}{ccc}
\text{a} & \text{b} & \text{c} \\
\text{b} & \text{d} & \text{e} \\
\text{c} & \text{e} & \text{f}
\end{array}
\]

Now, observe that state \(d\) is equivalent to state \(c\).

\[
\begin{array}{ccc}
\text{a} & \text{b} & \text{c} \\
\text{b} & \text{c} & \text{e} \\
\text{c} & \text{e} & \text{f}
\end{array}
\]

With the above states, one can write down the following first-step equations.

\[
\begin{align*}
T_a &= 1 + T_b \\
T_b &= 1 + \frac{1}{3}T_a + \frac{2}{3}T_c \\
T_c &= 1 + \frac{1}{2}T_b + \frac{1}{2}T_e \\
T_e &= 1 + \frac{2}{3}T_c + \frac{1}{3}T_f \\
T_f &= 0
\end{align*}
\]

Solving the above equations gives:

\[
T_a = 18, T_b = 17, T_c = 15, T_e = 11
\]

Thus, the player has to make 18 moves to go downstairs on average.

2. Consider 9 initial states and corresponding probabilities of reaching the “good” stairs as follows. Using symmetry, one can obtain the following table.

\[
\begin{array}{ccc}
p & \frac{1}{2} & 1 - p \\
q & \frac{1}{2} & 1 - q \\
0 & \frac{1}{2} & 1
\end{array}
\]
With the above probabilities, one can write down the following first-step equations.

\[ p = \frac{1}{2} q + \frac{1}{2} \cdot \frac{1}{2} \]
\[ q = \frac{1}{3} p + \frac{1}{3} \cdot \frac{1}{2} + \frac{1}{3} \cdot 0 \]

Solving the above equations gives:

\[ p = 0.4, q = 0.3 \]

Thus, we are going to reach the good stairs with probability 0.4.

Problem 5. Metropolis-Hastings Algorithm
In this problem we introduce the Metropolis-Hastings Algorithm, which is an example of Markov Chain Monte Carlo (MCMC) sampling. In the lab this week, you will implement Metropolis-Hastings and explore its performance.

Suppose that \( \pi \) is a probability distribution on a finite set \( X \). Assume that we can compute \( \pi \) up to a normalizing constant. Specifically, assume that we can efficiently calculate \( \tilde{\pi}(x) \) for any \( x \in X \), where \( \pi(x) = \tilde{\pi}(x)/\sum_{x' \in X} \tilde{\pi}(x') \). The normalizing constant \( 1/\sum_{x' \in X} \tilde{\pi}(x') \) is called the partition function in some contexts, and it can be difficult to compute if \( X \) is very large.

Instead of computing \( \pi \) directly, we will use \( \tilde{\pi} \) to design an algorithm to sample from the distribution \( \pi \). We can then approximate \( \pi \) if we take enough samples. The idea behind MCMC methods is to design a Markov chain whose stationary distribution is \( \pi \); then, we can “run” the chain until it is close to stationarity, and then collect samples from the chain.

Initialize the chain with \( X_0 = x_0 \), where \( x_0 \) is picked arbitrarily from \( X \). Let \( f : X \times X \to [0, 1] \) be a proposal distribution: for each \( x \in X \), \( f(x, \cdot) \) is a probability distribution on \( X \). (In the lab, you will look at what the desirable properties of a proposal distribution are.) If the chain is at state \( x \in X \), the chain makes a transition according to the following rule:

- Propose the next state \( y \) according to the distribution \( f(x, \cdot) \).
- Accept the proposal with probability

\[ A(x, y) = \min\left\{ 1, \frac{\pi(y)}{\pi(x)} \frac{f(y, x)}{f(x, y)} \right\}. \]

- If the proposal is accepted, then move the chain to \( y \); otherwise, stay at \( x \).

Assume that the proposal distribution \( f \) is chosen to make the chain irreducible.

1. Explain why the Markov chain can be simulated efficiently, even though \( \pi \) cannot be computed efficiently.
2. The key to showing why Metropolis-Hastings works is to look at the **detailed balance equations**. Suppose we have a finite irreducible Markov chain on a state space $X$ with transition matrix $P$. Show that if there exists a distribution $\pi$ on $X$ such that for all $x, y \in X$,

$$\pi(x)P(x, y) = \pi(y)P(y, x),$$

then $\pi$ is the stationary distribution of the chain. If these equations hold, then the Markov chain is called **reversible** because it turns out that the equations imply that the chain looks the same going forwards as backwards.

3. Now return to the Metropolis-Hastings chain. Use detailed balance to argue that $\pi$ is the stationary distribution of the chain.

4. If the chain is aperiodic, then the chain will converge to the stationary distribution. If the chain is not aperiodic, we can force it to be aperiodic by considering the **lazy chain**: on each transition, the chain decides not to move with probability $1/2$ (independently of the propose-accept step). Explain why the lazy chain is aperiodic, and explain why the stationary distribution is the same as before.

**Solution**

1. In order to simulate the Metropolis-Hastings chain, we need to draw samples from the proposal distribution, so the proposal distribution must be chosen to be efficiently computable. To compute the acceptance probability $A(x, y)$, observe that we only need the ratio

$$\frac{\pi(y)}{\pi(x)} = \frac{\tilde{\pi}(y)}{\tilde{\pi}(x)},$$

since the normalizing constant cancels out. Thus, we only need to compute $\tilde{\pi}$ in order to simulate the chain.

2. Suppose that detailed balance holds. Then,

$$\sum_{x \in X} \pi(x)P(x, y) = \sum_{x \in X} \pi(y)P(y, x) = \pi(y)\sum_{x \in X} P(y, x) = \pi(y),$$

so the balance equations hold, i.e., $\pi$ is the stationary distribution.

3. We will check that detailed balance holds for a pair of states $(x, y)$. Without loss of generality, assume that $\pi(y)f(y, x) \geq \pi(x)f(x, y)$. Then,

$$\pi(x)P(x, y) = \pi(x)f(x, y),$$

since our assumption means that the proposal $x \rightarrow y$ is always accepted. On the other hand,

$$\pi(y)P(y, x) = \pi(y)f(y, x)\frac{\pi(x)}{\pi(y)}\frac{f(x, y)}{f(y, x)} = \pi(x)f(x, y),$$

because the proposal $y \rightarrow x$ is accepted with probability

$$A(y, x) = \frac{\pi(x)f(x, y)}{\pi(y)f(y, x)}.$$
4. Now the chain has self-loops so it is aperiodic. The fact that the stationary distribution is unchanged can be argued because detailed balance still holds. Alternatively, the stationary distribution satisfies $\pi P = \pi$. The lazy chain is equivalent to replacing $P$ with $(P + I)/2$, where $I$ is the identity matrix, but then $\pi(P + I)/2 = \pi$, so $\pi$ is still stationary for the lazy chain.