1. Sub-Critical Forest

Consider a random graph $G(n,p(n))$ where $p(n) = o\left(\frac{1}{n}\right)$ (this is called the sub-critical phase). Show that the probability that $G(n,p(n))$ is a forest, i.e., contains no cycles, tends to 1 as $n \to \infty$.

*Hint:* Let $X_n$ be the number of cycles, and use Markov’s inequality.

2. Random Graph

Consider a random undirected graph on $n$ vertices, where each of the $\binom{n}{2}$ possible edges is present with probability $p$ independently of all the other edges. If $p = 0$ we have a fully empty graph with $n$ completely disconnected vertices; in contrast, if $p = 1$, every edge exists, the network is an $n$-clique, and every vertex is a distance one from every other vertex.

(a) Fix a particular vertex of the graph, and let $D$ be a random variable which is equal to the degree of this vertex. What is the PMF of $D$? Calculate $\lambda \triangleq \mathbb{E}[D]$.

(b) Assume that $c = np$ is a constant, independent of $n$. For large values of $n$, how would you approximate the PMF of $D$?

3. Exponential: MLE & MAP

The random variable $X$ is exponentially distributed with mean 1. Given $X$, the random variable $Y$ is exponentially distributed with rate $X$.

(a) Find MLE$[X \mid Y]$.

(b) Find MAP$[X \mid Y]$. 