1. Gaussians and the MSE
Suppose you draw \( n \) i.i.d. data points \((x_1, y_1), \ldots, (x_n, y_n)\), where \( n \) is a positive integer and the true relationship is \( Y = WX + \epsilon, \ \epsilon \sim \mathcal{N}(0, \sigma^2) \). (That is, \( Y \) has a linear dependence on \( X \), with additive Gaussian noise.) Show that finding the MLE estimate of \( W \) given the data points \( \{(x_i, y_i) : i = 1, \ldots, n\} \) is equivalent to minimizing the cost function

\[
    J(w) = \sum_{i=1}^{n}(y_i - wx_i)^2
\]

2. Hypothesis Testing for Bernoulli Random Variables
Assume that
- If \( X = 0 \), then \( Y \sim \text{Bernoulli}(1/4) \).
- If \( X = 1 \), then \( Y \sim \text{Bernoulli}(3/4) \).

Using the Neyman-Pearson formulation of hypothesis testing, find the optimal randomized decision rule \( r : \{0, 1\} \rightarrow \{0, 1\} \) with respect to the criterion

\[
    \min_{\text{randomized } r : \{0, 1\} \rightarrow \{0, 1\}} \mathbb{P}(r(Y) = 0 \mid X = 1) \quad \text{s.t. } \mathbb{P}(r(Y) = 1 \mid X = 0) \leq \beta,
\]

where \( \beta \in [0, 1] \) is a given upper bound on the false positive probability.

3. Bayesian Hypothesis Testing for Gaussian Distribution
Assume that \( X \) has prior probabilities \( \mathbb{P}(X = 0) = \mathbb{P}(X = 1) = 1/2 \). Further
- If \( X = 0 \), then \( Y \sim \mathcal{N}(\mu_0, \sigma_0^2) \).
- If \( X = 1 \), then \( Y \sim \mathcal{N}(\mu_1, \sigma_1^2) \).

Assume \( \mu_0 < \mu_1 \) and \( \sigma_0 < \sigma_1 \).

Using the Bayesian formulation of hypothesis testing, find the optimal decision rule \( r : \mathbb{R} \rightarrow \{0, 1\} \) with respect to the minimum expected cost criterion

\[
    \min_{r : \mathbb{R} \rightarrow \{0, 1\}} \mathbb{E}[I\{r(Y) \neq X\}].
\]