Midterm 2

Rules.

- No form of collaboration between the students is allowed. If you are caught cheating, you may fail the course and face disciplinary consequences.
- You have 10 minutes to read the exam and 90 minutes to complete it.
- The exam is not open book. No calculators or phones allowed.
- Unless otherwise stated, all your answers need to be justified.

<table>
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1 Assorted Problems [36]

(a) Bipartite Markov Chain [4]

Consider the following undirected bipartite Markov chain. At each time step, you pick one of your neighbors uniformly at random to transition to. If we start with an arbitrary distribution \( \pi_0 \), then the distribution of this Markov Chain after \( k \) transitions will always converge to the stationary distribution as \( k \to \infty \). Justify your answer.

\[
\begin{tikzpicture}
  \node[circle,draw] (L1) at (0,0) {};
  \node[circle,draw] (L2) at (0,-1) {};
  \node[circle,draw] (L3) at (0,-2) {};
  \node[circle,draw] (L4) at (0,-3) {};
  \node[circle,draw] (R1) at (2,0) {};
  \node[circle,draw] (R2) at (2,-1) {};
  \node[circle,draw] (R3) at (2,-2) {};
  \node[circle,draw] (R4) at (2,-3) {};
  \draw (L1) -- (R1);
  \draw (L1) -- (R2);
  \draw (L2) -- (R1);
  \draw (L2) -- (R2);
  \draw (L3) -- (R1);
  \draw (L3) -- (R2);
  \draw (L4) -- (R1);
  \draw (L4) -- (R2);
\end{tikzpicture}
\]

Figure 1: *

Complete Bipartite Graph for \( m = 3, n = 4 \)

\[ \text{\bigcirc True} \quad \text{\bigcirc False} \]

(b) School Cancellations [8]

In Berkeley, power outages happen according to a Poisson Process with a rate of \( \lambda_p \) and independently earthquakes happen according to a Poisson Process with a rate of \( \lambda_e \). Marc Fisher cancels school with probability \( p_p \) if there is a power outage, and \( p_e \) if there is an earthquake. What is the expectation and the variance of the amount of time \( T \) between the previous school cancellation and the next school cancellation from today? You may assume that this trend has been going on since infinitely in the past.

\[
E(T) =
\]

\[
\text{var}(T) =
\]
(c) **Entropy of a Markov Chain** [8]

Consider a random walk along the integers \(\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\). You start at 0 at time 0 and pick a direction (positive or negative with equal probability) to move in for time 1. At every time step after 1, you reverse direction with probability 0.25 and take a step in the new direction, or continue in the same direction otherwise. What is \(H(X_0, \ldots, X_{n-1})\)? You may leave your answer in terms of \(H_b(p) = -p \log p - (1 - p) \log(1 - p)\).

(d) **Poisson Process and Covariance** [8]

Consider a Poisson Process \(\{N(s) : s \in [0, \infty)\}\) with rate \(\lambda\).

Find the covariance of \(N(t_1)\) and \(N(t_2)\) for \(t_2 > t_1 \geq 0\).
(e) **Metropolis Hastings** [8]

Answer the following True/False questions about the MCMC lab. Briefly justify your answer in one or two sentences.

Metropolis Hastings allows us to generate random samples from a distribution \( p(x) \) even if it’s intractable to compute.

- True
- False

Any Metropolis Hastings Markov chain can be made aperiodic.

- True
- False

Burn in time is how long the Markov chain takes to converge to the state whose stationary distribution probability is maximal.

- True
- False

In the Traveling Salesman Problem, you want to find the shortest path to visit \( n \) cities. Even though this problem has \( n! \) possible answers and is NP-hard, we can approximately solve it with Metropolis Hastings. Our plan is to design a Markov chain such that if \( x^* \) is the best path, the stationary distribution probability \( \pi(x^*) \) will be maximal. Then by running the Markov chain for a long time and looking at the most commonly visited state, we can infer the best path. If \( L(x) \) is the length of a path \( x \), then for this problem we could let our directly proportional estimate \( f(x) \) be equal to \( L(x) \).

- True
- False
2 Convergence [20]

(a) Convergent Balls [10]

Kevin has a basket of $k$ balls. Each ball is either white or red. Initially, all of the balls in his basket is red. Let $X_i^{(n)}$ denote the if the $i$-th ball is red at time step $n$. And let $Y_n = \sum_{i=1}^{k} X_i^{(n)}$ therefore be the number of balls in Kevin’s basket that are red at time step $n$. At every time step, Kevin takes a ball out uniformly at random, and he replaces the ball with a white ball. Prove that $Y_n$ converges in probability to 0.

(b) Markov Chain Groups [10]

Consider the following Markov Chain

![Markov Chain Diagram]

Given a finite length sequence $x_1, \ldots, x_n$ sampled from this Markov chain, we can form groups for samples that are in the same state. For example, if our sequence is

\[
\underbrace{A, A, A}_1, \underbrace{B, B}_2, \underbrace{A, A}_3, \underbrace{B, B, B, B}_4
\]

Then we have 4 groups of size 3, 2, 2, 5. Let $G_n$ be

\[
G_n = \frac{n}{\text{number of groups among } x_1, \ldots, x_n}
\]

In the example, $G_{3+2+2+5} = G_{12} = \frac{12}{4} = 3$. What does $G_n$ converge to?
3 Discrete Time Markov Chains [30]

Consider the following Discrete Time Markov Chain.

(a) **Stationary Distribution** [10]

Find the stationary distribution of the chain.
(b) **Hitting Time Backwards [10]**
What is the expected hitting time from state $n$ to state 0?

(c) **Hitting Time Forward [10]**
Suppose $p = q = \frac{1}{2}$. What is the expected hitting time from state 0 to state $n$?

*Hint:* Use parts (a) and (b).
4 Last Arrival [26]

Suppose we have two independent Poisson Processes $X$ and $Y$ with arrival rates $\lambda$ and $2\lambda$, respectively.

(a) **X or Y? [6]**

What is the probability that the last arrival for $Y$ comes after the last arrival for $X$ in some given time interval $(0, t)$?

(b) **Expected Arrivals Afterwards [10]**

Suppose we are given that $N_Y(t) = n$. What is the expected number of arrivals $N$ from $X$ after the last arrival from $Y$ on $(0, t)$?
(c) Chernoff Bound [10]

Provide an upper bound using the Chernoff bound on the probability that the total number of arrivals from \( X \) and \( Y \) in the interval \((0, t)\) exceed \( k\lambda t \) for some positive integer \( k \). Express your answer in terms of \( k \) and simplify your answer as much as possible.

Hint: The MGF for a Poisson random variable \( X \) with parameter \( \lambda \) is \( M_X(s) = \exp(\lambda(e^s - 1)) \).
5 Continuous Time Markov Chains [14]

Consider the following CTMC for $r > 0$.

(a) **Stationary Distribution** [7]

For the above CTMC, find the stationary distribution in terms of $r$, where $r > 0$.

(b) **Equivalent DTMC** [7]

Draw a DTMC with the same state space with the same stationary distribution as the above CTMC. If necessary, draw multiple DTMCs depending on the value of $r$. 