Problem 1. Detection, repetition coding and beyond

In class we have talked about the detection problem in which random signal $X = 1, 0$ passes through a BSC channel with crossover probability ε and we observes Y. Now suppose $P(X = 0) = p = 0.3$ and $\varepsilon = 0.4$.

a) We have the freedom to design a decoder which has the observation Y as input and a detected $\hat{X}$ as output. We use the probability error $p_e = P(X \neq \hat{X})$ as the criteria. What is the optimal detection rule, given the criteria? What is the minimum p_e?

b) In reality, we also have the freedom to design an encoder which has X as input and output $\bar{X}$ to transmit over the BSC channel. Now suppose we are using repetition coding, i.e. transmit $\bar{X} = 111$ if $X = 1$ and $\bar{X} = 000$ if $X = 0$. As a result, the Y we observe is now a three-bits sequence. How to do the optimal detection? What is the minimum p_e? What is the p_e given observing $Y = 111$?

c) Now instead of decoding the $\hat{X}$ after we receive all three bits, we decode it on fly, i.e. decode $\hat{X}$ when we receive the first, second and third bits respectively. The decoding performance should be increasing as we get more observations (remember the Baysian learning stuff in homework and discussion before?). This exercise would give you some simple principle on how to improve performance of communication system. How to do the optimal on-fly decoding bit by bit? How the decoded $\hat{X}$ changes as we get more and more observation? For the bits we observed is '111', compare the p_e on receiving three bits with the one in part b). What is the advantage of on-fly decoding?

d) In reality, people are using the simple principle to design a complex while powerful code, named TURBO code[1]. It has the ability to almost achieve the performance limit computed by Shannon back to 1948. The on-fly decoding idea in part c) is also used in many online detection/estimation problem, one can refer to Viterbi decoder[2] as a concrete example.

[1] Please visit http://www331.jpl.nasa.gov/public/JPLtcodes.html to get more details, if you are interested in it.

[2] You might need to look into any textbook on digital communications to find the details.