EE 126: Probability and Random Processes

Problem Set 2

Due: February 5 in class

Problems 17, 24 in http://www.athenasc.com/CH1-prob-supp.pdf.

3. A laboratory blood test is 95 percent effective in detecting a certain disease when it is in fact present. However, the test also yields a "false positive" result for 1 percent of the healthy person tested. (That is, if a healthy person is tested, then with probability 0.01, the test will imply that he has the disease.) If 0.5 percent of the population actually has the disease, what is the probability a person has the disease given that his test result is positive? Are you surprised by the answer? Explain.

4. (a) We are given a sample space Ω and a probability law P. Let B be an event with non-zero probability. Define the real-valued function Q on the set of events in Ω by $Q(A) := P(A|B)$. Show that Q satisfies the three axioms of a probability law and hence is a valid probability law.

b) Using (a) or by direct calculations, show the formula we discussed in class:

$$P(A|B,C) = \frac{P(C|A,B)P(A|B)}{P(C|B)} \tag{1}$$

(Hint: express everything in terms of the probability law Q defined in (a).)

5. (Bayesian learning.)

(a) We transmit a bit of information which is 0 with probability p and 1 with $1-p$. It passes through a BSC with cross-over probability ϵ. Suppose we observe a 1 at the output. Find the conditional probability p_1 that the transmitted bit is a 1.

(b) The same bit is transmitted again through the BSC and you observe another 1. Find a formula to update p_1 to get p_2, the conditional probability that the transmitted bit is a 1. (You may find eqn. (1 useful.)

(c) Using (b) or otherwise, calculate p_n, the probability that the transmitted bit is a 1 given that you have observed n 1's at the BSC output. Plot p_n as a function of n. What happens as $n \to \infty$?

(d) You declare that the transmitted bit is a 1 whenever p_n exceeds 0.99. How long do you have to wait? How does your answer qualitatively depend on p and ϵ? Does it make intuitive sense? Explain.

6. Let's consider the example in class where the document has n letters and the probability that the kth letter is typed wrongly is p.

(a) Explain why there is not enough information to calculate the probability that the entire document is typed correctly.

(b) Find the smallest possible value for this probability in terms of p and specify the probability law for which this smallest value is achieved.

(c) Let $n = 2$. Suppose you are told additionally that q_1 is the conditional probability that the second letter is typed wrongly given that the first letter is typed wrongly, and q_2 be the conditional probability that the second letter is typed wrongly given that the first letter is typed correctly. Do we now have a complete specification of the probability law?

(d) What constraint does q_1 and q_2 have to satisfy?

e) Under what values of q_1 and q_2 is the probability that the document is typed correctly (i) equal to, (ii) larger than, and (iii) smaller than the probability when the events that the two letters are typed incorrectly are independent?