EECS 126: Probability and Random Processes
Problem Set 11
Due on Thursday, April 28th 2005 in class

Problem 20. [D] Arrivals of certain events at points in time are known to constitute a Poisson process, but it is not known which of two possible values of λ, the average arrival rate, describes the process. Our a priori estimate is that $\lambda = 2$ or $\lambda = 4$ with equal probability. We observe the process for t units of time and observe exactly k arrivals. Given this information, determine the conditional probability that $\lambda = 2$. Check to see whether or not your answer is reasonable for some simple limiting values for k and t.

Problem 21. [D] Let $K_1, , K_2, \ldots$ be independent identically distributed geometric random variables. Random variable R_i is defined by

$$R_i = \sum_{j=1}^{i} K_i, \qquad i = 1, 2, \ldots.$$

If we eliminate arrivals number $R_1, R_2, \ldots$ in a Poisson process, do the remaining arrivals constitute a Poisson process?

Problem 25. There are two types of calls to the MIT Campus Patrol. Type A calls (distress calls) arrive as a Poisson process with rate λ_A. Type B calls (professors who have lost their keys) arrive as an independent Poisson process with rate λ_B. Let us fix t to be 12 o'clock.

(a) What is the expected length of the interval that t belongs to? (That is, the interval from the last event before t until the first event after t.)

(b) What is the probability that t belongs to an AA interval? (That is, the first event before, as well as the first event after time t are both of type A.)

(c) Let c be a constant. What is the probability that between t and $t + c$, we have exactly two events, one of type A, followed by one of type B?

Problem 27.

(a) Shuttles depart from New York to Boston every hour on the hour. Passengers arrive according to a Poisson process of rate λ per hour. Find the expected number of passengers on a shuttle. (Ignore issues of limited seating.)

(b) Now, and for the rest of this problem, suppose that the shuttles are not operating on a deterministic schedule, but rather their interdeparture times are exponentially distributed with rate μ per hour, and independent of the process of passenger arrivals. Find the PMF of the number shuttle departures in one hour.

(c) Let us define an "event" in the airport to be either the arrival of a passenger, or the departure of a plane. Find the expected number of "events" that occur in one hour.

(d) If a passenger arrives at the gate, and sees 2λ people waiting, find his/her expected time to wait until the next shuttle.

(e) Find the PMF of the number of people on a shuttle.

Problem 28. Type A, B, and C items are placed in a common buffer, each type arriving as part of an independent Poisson process with average arrival rates, respectively, of a, b, and c items per minute.

For the first four parts of this problem, assume the buffer is discharged immediately whenever it contains a total of ten items.

(a) What is the probability that, of the first ten items to arrive at the buffer, only the first and one other are type A?

(b) What is the probability that any particular discharge of the buffer contains five times as many type A items as type B items?

(c) Determine the PDF, expectation, and variance of the total time between consecutive discharges of the buffer.

(d) Determine the probability that during a particular five minute interval there exactly two arrivals of each type.

For the rest of this problem, a different rule is used for discharging the buffer: namely, the buffer is discharged immediately whenever it contains a total of three type A items.

Problem 30. Let Y be exponentially distributed with parameter λ_1. Let Z_k be Erlang of order k. with parameter λ_2. Assume that Y and Z_k are independent. Let $M_k = \max\{Y, Z_k\}$. Find a recursive formula for $\mathbf{E}[M_k]$, in terms of $E[M_{k-1}]$.

Problem 34. We are given the following statistics about the number of children in a typical family in a small village. There are 100 families. 10 families have no children, 40 have 1, 30 have 2, 10 have 3, 10 have 4.

(a) If you pick a family at random, what is the expected number of children in that family?

(b) If you pick a child at random (each child is equally likely), what is the expected number of children in that child's family (including the picked child)?

(c) Generalize your approach from part (b) to the case where a fraction p_k of the families has k children, and provide a formula.