EECS 126: Probability and Random Processes

Problem Set 3

Due on Thursday, February 17th 2005 in class

Problem 12. The MIT football team wins any one game with probability p, and loses it with probability $1-p$. Its performance in each game is independent of its performance in other games. Let L_1 be the number of losses before its first win, and let L_2 be the number of losses after its first win and before its second win. Find the joint PMF of L_1 and L_2.

Problem 16. Suppose that X and Y are independent discrete random variables with the same geometric PMF:

$$p_X(k) = p_Y(k) = p(1-p)^{k-1}, \qquad k = 1, 2, \ldots,$$

where p is a scalar with $0 < p < 1$. Show that for any integer $n \geq 2$, the conditional PMF

$$\mathbf{P}(X = k \,|\, X + Y = n)$$

is uniform.

Problem 17. Let X, Y, and Z be independent geometric random variables with the same PMF:

$$p_X(k) = p_Y(k) = p_Z(k) = p(1-p)^{k-1},$$

where p is a scalar with $0 < p < 1$. Find $\mathbf{P}(X = k \,|\, X+Y+Z = n)$. *Hint:* Try thinking in terms of coin tosses.

Problem 19. At his workplace, the first thing Oscar does every morning is to go to the supply room and pick up one, two, or three pens with equal probability 1/3. If he picks up three pens, he does not return to the supply room again that day. If he picks up one or two pens, he will make one additional trip to the supply room, where he again will pick up one, two, or three pens with equal probability 1/3. (The number of pens taken in one trip will not affect the number of pens taken in any other trip.) Calculate the following:

(a) The probability that Oscar gets a total of three pens on any particular day.

(b) The conditional probability that he visited the supply room twice on a given day, given that it is a day in which he got a total of three pens.

(c) $\mathbf{E}[N]$ and $\mathbf{E}[N \,|\, C]$, where $\mathbf{E}[N]$ is the unconditional expectation of N, the total number of pens Oscar gets on any given day, and $\mathbf{E}[N \,|\, C]$ is the conditional expectation of N given the event $C = \{N > 3\}$.

(d) $\sigma_{N \,|\, C}$, the conditional standard deviation of the total number of pens Oscar gets on a particular day, where N and C are as in part (c).

(e) The probability that he gets more than three pens on each of the next 16 days.

(f) The conditional standard deviation of the total number of pens he gets in the next 16 days given that he gets more than three pens on each of those days.

Problem 21. Joe Lucky plays the lottery on any given week with probability p, independently of whether he played on any other week. Each time he plays, he has a probability q of winning, again independently of everything else. During a fixed time period of n weeks, let X be the number of weeks that he played the lottery and Y be the number of weeks that he won.

(a) What is the probability that he played the lottery on any particular week, given that he did not win on that week?

(b) Find the conditional PMF $p_{Y\,|\,X}(y\,|\,x)$.

(c) Find the joint PMF $p_{X,Y}(x, y)$.

(d) Find the marginal PMF $p_Y(y)$. *Hint:* One possibility is to start with the answer to part (c), but the algebra can be messy. But if you think intuitively about the procedure that generates Y, you may be able to guess the answer.

(e) Find the conditional PMF $p_{X\,|\,Y}(x\,|\,y)$. Do this algebraically using the preceding answers.

(f) Rederive the answer to part (e) by thinking as follows: for each one of the $n - Y$ weeks that he did not win, the answer to part (a) should tell you something.

Problem 2. Let X be a random variable with PDF

$$f_X(x) = \begin{cases} 2x/3 & \text{if } 1 < x \le 2, \\ 0 & \text{otherwise,} \end{cases}$$

and let $Y = X^2$. Calculate $\mathbf{E}[Y]$ and $\text{var}(Y)$.

Problem 3. Find the PDF, the mean, and the variance of the random variable X with CDF

$$F_X(x) = \begin{cases} 1 - \dfrac{a^3}{x^3}, & \text{if } x \ge a, \\ 0, & \text{if } x < a, \end{cases}$$

where a is a positive constant.

Problem 5. The **median** of a random variable X is a number μ that satisfies $F_X(\mu) = 1/2$. Find the median of the exponential random variable with parameter λ.