EECS 126: Probability and Random Processes

Problem Set 5

Due on Thursday, February 24th 2005 in class

Problem 7. Let X be normal with mean 1 and variance 4. Let $Y = 2X + 3$.

(a) Calculate the PDF of Y.

(b) Find $\mathbf{P}(Y \geq 0)$.

Problem 8. A signal of amplitude $s = 2$ is transmitted from a satellite but is corrupted by noise, and the received signal is $Z = s + W$, where W is noise. When the weather is good, W is normal with zero mean and variance 1. When the weather is bad, W is normal with zero mean and variance 4. Good and bad weather are equally likely. In the absence of any weather information:

(a) Calculate the PDF of X.

(b) Calculate the probability that X is between 1 and 3.

Problem 9. Oscar uses his high-speed modem to connect to the internet. The modem transmits zeros and ones by sending signals -1 and $+1$, respectively. We assume that any given bit has probability p of being a zero. The telephone line introduces additive zero-mean Gaussian (normal) noise with variance σ^2 (so, the receiver at the other end receives a signal which is the sum of the transmitted signal and the channel noise). The value of the noise is assumed to be independent of the encoded signal value.

(a) Let a be a constant between -1 and 1. The receiver at the other end decides that the signal -1 (respectively, $+1$) was transmitted if the value it receives is less (respectively, more) than a. Find a formula for the probability of making an error.

(b) Find a numerical answer for the question of part (a) assuming that $p = 2/5$, $a = 1/2$ and $\sigma^2 = 1/4$.

Problem 11. Consider a random variable X with PDF

$$f_X(x) = \begin{cases} 2x/3, & \text{if } 1 < x \leq 2, \\ 0, & \text{otherwise,} \end{cases}$$

and let A be the event $\{X \geq 1.5\}$. Calculate $\mathbf{E}[X]$, $\mathbf{P}(A)$, and $\mathbf{E}[X \mid A]$.

Problem 13. One of two wheels of fortune, A and B, is selected by the toss of a fair coin, and the wheel chosen is spun once to determine the value of a random variable X. If wheel A is selected, the PDF of X is

$$f_{X|A}(x|A) = \begin{cases} 1 & \text{if } 0 < x \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

If wheel B is selected, the PDF of X is

$$f_{X|B}(x|B) = \begin{cases} 3 & \text{if } 0 < w \leq 1/3, \\ 0 & \text{otherwise.} \end{cases}$$

If we are told that the value of X was less than $1/4$, what is the conditional probability that wheel A was the one selected?

Problem 15. A family has three children, A, B, and C, of height X_1, X_2, X_3, respectively. If X_1, X_2, X_3 are independent and identically distributed continuous random variables, evaluate the following probabilities:

(a) $\mathbf{P}(A \text{ is the tallest child})$.

(b) $\mathbf{P}(A \text{ is taller than } B \mid A \text{ is taller than } C)$.

(c) $\mathbf{P}(A \text{ is taller than } B \mid B \text{ is taller than } C)$.

(d) $\mathbf{P}(A \text{ is taller than } B \mid A \text{ is shorter than } C)$.

(e) $\mathbf{P}(A \text{ is taller than } B \mid B \text{ is shorter than } C)$.

Problem 17. Let X and Y be independent random variables, with each one uniformly distributed in the interval $[0, 1]$. Find the probability of each of the following events.

(a) $X > 6/10$.

(b) $Y < X$.

(c) $X + Y \leq 3/10$.

(d) $\max\{X, Y\} \geq 1/3$.

(e) $XY \leq 1/4$.

Problem 21. The random variables X and Y have the joint PDF

$$f_{X,Y}(x, y) = \begin{cases} 2, & \text{if } x > 0 \text{ and } y > 0 \text{ and } x + y \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

Let A be the event $\{Y \leq 0.5\}$ and let B be the event $\{Y > X\}$.

(a) Calculate $\mathbf{P}(B \mid A)$.

(b) Calculate $f_{X|Y}(x \mid 0.5)$. Calculate also the conditional expectation and the conditional variance of X, given that $Y = 0.5$.

(c) Calculate $f_{X|B}(x)$.

(d) Calculate $\mathbf{E}[XY]$.

(e) Calculate the PDF of Y/X.