**EECS 126: Probability and Random Processes**
Problem Set 7
Due on Thursday, March 10th 2005 in class

**Problem 23.** Let $X$ be a geometric random variable with parameter $P$, where $P$ is itself random and uniformly distributed from 0 to $(n-1)/n$. Let $Z = \mathbf{E}[X \mid P]$. Find $\mathbf{E}[Z]$ and $\lim_{n\to\infty} \mathbf{E}[Z]$.

**Problem 2.**

(a) Given the information $\mathbf{E}[X] = 7$ and $\text{var}(X) = 9$, use the Chebyshev inequality to find a lower bound for the probability $\mathbf{P}(4 < X < 10)$.

(b) Find the smallest and the largest possible values of the probability $\mathbf{P}(4 < X < 10)$, given the mean and variance information from part (a).

**Problem 3.** Investigate whether the Chebyshev inequality is tight. That is, for every $\mu$, $\sigma$, and $c \geq \sigma$, does there exist a random variable $X$ with mean $\mu$ and standard deviation $\sigma$ such that

$$\mathbf{P}\Big(|X-\mu| \geq c\Big) = \frac{\sigma^2}{c^2}?$$

**Problem 5.** Bo assumes that $X$, the height in meters of any Canadian selected by an equally likely choice among all Canadians, is a random variable with $\mathbf{E}[X] = h$. Because Bo is sure that no Canadian is taller than 3 meters, he decides to use 1.5 meters as a conservative value for the standard deviation of $X$. To estimate $h$, Bo uses the average of the heights of $n$ Canadians he selects at random.

(a) In terms of $h$ and Bo's 1.5 meter bound for the standard deviation of $X$, determine the expectation and standard deviation of $H$.

(b) Find as small a value of $n$ as possible such that the standard deviation of Bo's estimator is guaranteed to be less than 0.01 meters.

(c) Bo would like to be 99% sure that his estimate is within 5 centimeters of the true average height of Canadians. Using the Chebyshev inequality, calculate the minimum value of $n$ that will achieve this objective.

(d) If we agree that no Canadians are taller than three meters, why is it correct to use 1.5 meters as an upper bound on the standard deviation for $X$, the height of any Canadian selected at random?

**Problem 9.** Let $X_1, X_2, \ldots$ be independent, identically distributed random variables with (unknown but finite) mean $\mu$ and positive variance. For $i = 1, 2, \ldots$, let

$$Y_i = \frac{1}{3}X_i + \frac{2}{3}X_{i+1}.$$

(a) Are the random variables $Y_i$ independent?

(b) Are they identically distributed?

(c) Let

$$M_n = \frac{1}{n}\sum_{i=1}^{n} Y_i.$$

Show that $M_n$ converges to $\mu$ in probability.

**Problem 10.** Let $X_1, X_2, \ldots$ be a sequence of independent random variables that are uniformly distributed between 0 and 1. For every $n$, we let $Y_n$ be the median of the values of $X_1, X_2, \ldots, X_{2n+1}$. [That is, we order $X_1, \ldots, X_{2n+1}$ in increasing order and let $Y_n$ be the $(n+1)$st element in this ordered sequence.] Show that that the sequence $Y_n$ converges to $1/2$, in probability.

**Problem 14.** Let $X_1, \ldots, X_{10}$ be independent random variables, uniformly distributed over the unit interval $[0, 1]$.

(a) Estimate $\mathbf{P}(X_1 + \cdots + X_{10} \geq 7)$ using the Markov inequality.

(b) Repeat part (a) using the Chebyshev inequality.

(c) Repeat part (a) using the central limit theorem.

(d) Repeat part (a) using the Chernoff bound

(e) Calculate numerically part (a) using a computer