**EECS 126: Probability and Random Processes**
Problem Set 8
Due on Thursday, March 17th 2005 in class

**Problem 6.** Let $X_1, X_2, \ldots$ be independent, identically distributed random variables with $\mathbf{E}[X] = 2$ and var($X$)=9, and let $Y_i = X_i/2^i$. We also define $T_n$ and $A_n$ to be the sum and the sample mean, respectively, of the random variables $Y_1, \ldots, Y_n$.

(a) Evaluate the mean and variance of $Y_n$, $T_n$, and $A_n$.

(b) Does $Y_n$ converge in probability? If so, to what value?

(c) Does $T_n$ converge in probability? If so, to what value?

(d) Does $A_n$ converge in probability? If so, to what value?

**Problem 7.** Suppose that a sequence $X_n$ of random variables satisfies

$$\lim_{n\to\infty} \mathbf{E}\big[|X_n - c|^\alpha\big] = 0,$$

where $\alpha$ is a positive number. Show that the sequence $X_n$ converges to $c$ in probability.

**Problem 12.** On any given flight, an airline's goal is to fill the plane as much as possible, without overbooking. If, on average, 10% of customers cancel their tickets, all independently of each other, what is the probability that a particular flight will be overbooked if the airline sells 320 tickets, for a plane that has maximum capacity 300 people? What is the probability that a plane with maximum capacity 150 people will be overbooked if the airline sells 160 tickets?

**Problem 15.** Let $S_n$ be the number of successes in $n$ independent Bernoulli trials, where the probability of success in each trial is $p = 1/2$. Provide a numerical value for the limit as $n$ tends to infinity for each of the following three expressions.

(a) $\mathbf{P}\big(\frac{n}{2} - 10 \le S_n \le \frac{n}{2} + 10\big)$.

(b) $\mathbf{P}\big(\frac{n}{2} - \frac{n}{10} \le S_n \le \frac{n}{2} + \frac{n}{10}\big)$.

(c) $\mathbf{P}\big(\frac{n}{2} - \frac{\sqrt{n}}{2} \le S_n \le \frac{n}{2} + \frac{\sqrt{n}}{2}\big)$.

**Problem 17.** Let $X_1, X_2, \ldots$ be a sequence of independent identically distributed random variables with finite mean and variance. Show that the sequence $Y_n = X_n/n$ converges to zero, with probability 1.

**Problem 18.** Let $X_1, X_2, \ldots$ be independent identically distributed random variables with mean 5, variance 9, and such that $\mathbf{P}(X_n = 0) > 0$. For each of the following ways of defining $Z_n$, determine whether the sequence $Z_n$ converges with probability 1, and if it does, identify the limit.

(a) $Z_n = (X_1 + \cdots + X_n)/n$.

(b) $Z_n = (X_1 + \cdots + X_n - 5n)/\sqrt{n}$.

(c) $Z_n = (X_1^2 + \cdots + X_n^2)/n$.

(d) $Z_n = X_1 X_2 \cdots X_n$.

(e) $Z_n = (X_1 X_2 + X_2 X_3 + \cdots + X_{n-1} X_n)/n$.

**Problem 19.** The fortune $X_n$ of a gambler evolves as $X_n = Z_n X_{n-1}$, where the $Z_n$ are independent identically distributed random variables with PMF

$$p_Z(z) = \begin{cases} 1/3, & \text{for } z = 3, \\ 2/3, & \text{for } z = 1/3. \end{cases}$$

Assume that $X_0 = 1$.

(a) Show that the expected fortune $\mathbf{E}[X_n]$ converges to infinity as $n$ increases.

(b) Show that the actual fortune $X_n$ converges to 0 with probability 1.