EECS 126: Probability and Random Processes
Problem Set 9
Due on Thursday, April 7th 2005 in class

Problem 32. Let $X_1, \ldots, X_n$ be some random variables and let $c_{ij} = \mathrm{cov}(X_i, X_j)$. Show that for any numbers $a_1, \ldots, a_n$, we have

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_i c_{ij} a_j \geq 0.$$

Problem 33. Consider n independent tosses of a die. Each toss has probability p_i of resulting in i. Let X_i be the number of tosses that result in i. Show that X_1 and X_2 are negatively correlated (i.e., a large number of ones suggests a smaller number of twos).

Problem 34. Let $X = Y - Z$ where Y and Z are nonnegative random variables such that $YZ = 0$.

(a) Show that $\mathrm{cov}(Y, Z) \leq 0$.

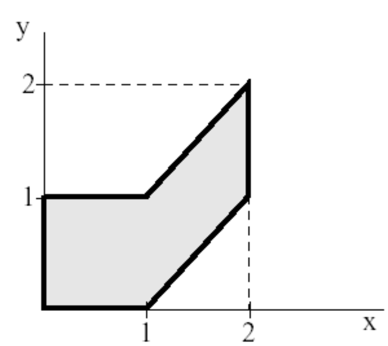
(b) Show that $\mathrm{var}(X) \geq \mathrm{var}(Y) + \mathrm{var}(Z)$.

(c) Use the result of part (b) to show that

$$\mathrm{var}(X) \geq \mathrm{var}\big(\max\{0, X\}\big) + \mathrm{var}\big(\max\{0, -X\}\big).$$

Problem 36. The continuous random variables X and Y have a joint PDF given by

$$f_{X,Y}(x, y) = \begin{cases} c, & \text{if } (x, y) \text{ belongs to the shaded region} \\ 0, & \text{otherwise.} \end{cases}$$



(a) Find the least squares estimate of Y given that $X = x$, for all possible vaalues x.

(b) Let $g^*(x)$ be the estimate from part (a), as a function of x. Find $\mathbf{E}[g^*(X)]$ and $\text{var}(g^*(X))$.

(c) Find the mean square error $\mathbf{E}[(Y - g^*(X))^2]$. Is it the same as $\mathbf{E}[\text{var}(Y \mid X)]$?

(d) Find $\text{var}(Y)$.

Problem 38. In a communication system, the value of a random variable X is transmitted, but what is received (denoted by Y) is the value of X corrupted by some additive noise W; that is, $Y = X + W$. We know the distribution of X and W, and let us assume that these two random variables are independent and have the same PDF. Calculate the least squares estimate of X given Y. What happens if X and W are dependent?

Problem 40. Provide a new derivation of the formula

$$\text{var}(X) = \text{var}(\hat{X}) + \text{var}(\tilde{X})$$

using the law of total variance. Here, as in the text, $\hat{X} = \mathbf{E}[X \mid Y]$, and $\tilde{X} = X - \hat{X}$.

Problem 41. Let U and V be independent standard normal random variables. Let

$$X = U + V, \qquad Y = U - 2V.$$

(a) Do X and Y have a bivariate normal distribution?

(b) Provide a formula for $\mathbf{E}[X \mid Y]$.

(c) Write down the joint PDF of X and Y.