

Spring 2005: EECS126 Midterm 2 **SOLUTIONS**  
*No Collaboration Permitted*

**Write your name and SID on every sheet of the exam**

Be as clear and precise in your answers if possible

Come to the front if you have a question.

*You do not have to do every part of every problem to get full credit on the exam. Doing more will give you extra credit.*

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**Problem 2.1** (42pts) True or False. Prove or show a counterexample:

a. 14pts. If  $X$  and  $Y$  are rv's, then:  $\text{var}(X|Y = y) \leq \text{var}(X)$  regardless of what value  $y$  is.

**FALSE.**

Let  $Y, Z$  be Bernoulli ( $\frac{1}{2}$ )

let  $X = Y \cdot Z \Rightarrow$  Bernoulli ( $\frac{1}{4}$ )

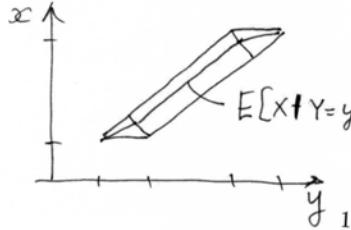
$$\text{Var}(X) = \frac{3}{16}$$

However,  $\text{var}(X|Y=1) = \text{var}(Z) = \frac{1}{4} > \frac{3}{16} = \text{var}(X)$

b. 14pts The linear least squares estimate of a rv.  $X$  given rv.  $Y = y$  is the same as  $E[X|Y = y]$ .  
**FALSE.**

Linear least squares estimate is always linear function of observation, while  $E[X|Y=y]$  is not necessarily.

Remember example 4.27 in the textbook



$E[x|Y=y] \rightarrow$  piecewise linear  
 while  $\text{LLSE}[x|Y=y]$   
 is linear, regardless  
 what the value  $y$  is!

- c. 14pts If  $X, Y$  are jointly Normal random variables then the conditional density  $f_{X|Y}(x|y)$  is always Normal regardless of what the value  $y$  is.

TRUE.

$X, Y$  jointly Gaussian, there exists  $a, b, c, d$  and  $U, V$  i.i.d.  $\sim N(0)$

$$\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} U \\ V \end{bmatrix} \quad \begin{aligned} X &= aU + bV \\ Y &= cU + dV \end{aligned}$$

Note that  $X$  can be expressed as

$X = gY + Z$  where  $Z$  uncorrelated with  $Y$   $E[Z^2] = 0$   
i.e. independent since Gaussian

$$E[X] = gE[Y] + E[Z] \Rightarrow E[Z] = 0.$$

$$E[X^2] = g^2 E[Y^2] + E[Z^2]$$

$$E[XY] = gE[Y^2]$$

$$E[XY] = ac + bd$$

$$E[X^2] = a^2 + b^2$$

$$E[Y^2] = c^2 + d^2$$

$$\text{Therefore, } g = \frac{ac+bd}{c^2+d^2} \quad \text{and } E[Z^2] = E[X^2] - g^2 E[Y^2]$$

$$X = \frac{ac+bd}{c^2+d^2} \cdot Y + Z$$

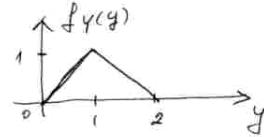
so given  $Y=y$ ,  $X$  is still Gaussian  
 $N(0, E[X^2] - gE[Y^2])$ .

**Problem 2.2 (36pts)** Let  $X_1$  and  $X_2$  be two independent continuous uniform random variables Uniform[0, 1].

a. 6pts. Derive CDF and PDF of  $Y = X_1 + X_2$ .

$$Y = X_1 + X_2, \quad X_1, X_2 \text{ independent}$$

$$f_Y(y) = \int_{-\infty}^{+\infty} f_{X_1}(t) \cdot f_{X_2}(y-t) dt$$



$$= \begin{cases} 0 & y \leq 0 \\ \int_0^y 1 dt & 0 \leq y \leq 1 \\ \int_{y-1}^1 1 dt & 1 \leq y \leq 2 \\ 0 & y > 2 \end{cases} = \begin{cases} 0 & y \leq 0 \\ y & 0 \leq y \leq 1 \\ 2-y & 1 \leq y \leq 2 \\ 0 & y > 2 \end{cases}$$

$$F_Y(y) = \int_{-\infty}^y f_Y(t) dt = \begin{cases} 0 & y \leq 0 \\ y^2/2 & 0 \leq y \leq 1 \\ -\frac{1}{2}y^2 + 2y - 1 & 1 \leq y \leq 2 \\ 1 & y > 2 \end{cases}$$

b. 6pts. Derive CDF and PDF of  $Z = \max\{X_1, X_2\}$ .

$$F_Z(z) = P(X_1 \leq z, X_2 \leq z) = P(X_1 \leq z) \cdot P(X_2 \leq z)$$

$$= z^2 \quad \text{for } 0 \leq z \leq 1$$

$$F_Z(z) = \begin{cases} 0 & \text{if } z < 0 \\ z^2 & \text{if } 0 \leq z \leq 1 \\ 1 & \text{if } z > 1 \end{cases}$$

$$f_Z(z) = \frac{d}{dz} F_Z(z) = \begin{cases} 2z & \text{if } 0 \leq z \leq 1 \\ 0 & \text{o.w.} \end{cases}$$

c. 6pts. Derive conditional CDF  $F_{X_1|Y}(x_1|y)$  and conditional PDF  $f_{X_1|Y}(x_1|y)$ .

$$f_{X_1|Y}(x_1|y) = \frac{f_{X_1,Y}(x_1,y)}{f_Y(y)} = \frac{f_{X_1}(x_1) \cdot f_{Y|X_1}(y|x_1)}{f_Y(y)} \text{ solve a) first.}$$

$$f_{X_1,Y}(x_1,y) = \begin{cases} \frac{1}{y} & \text{if } y \in [0,1] \text{ and } x_1 \in [0,y] \\ \frac{1}{2-y} & \text{if } y \in [1,2] \text{ and } x_1 \in [y-1,1] \end{cases}$$

$$F_{X_1|Y}(x_1|y) = \begin{cases} \int_0^{x_1} \frac{1}{t} dt = \frac{x_1}{y} & \text{if } y \in [0,1] \text{ and } x_1 \in [0,y] \\ \int_{y-1}^{x_1} \frac{1}{2-y} dt = \frac{x_1+1-y}{2-y} & \text{if } y \in [1,2] \text{ and } x_1 \in [y-1,y] \end{cases}$$

d. 6pts. Derive conditional CDF  $F_{Y|X_1}(y|x_1)$  and conditional PDF  $f_{Y|X_1}(y|x_1)$ .

$$\begin{aligned} F_{Y|X_1}(y|x_1) &= P(Y \leq y | X_1 = x_1) = P(X_1 + X_2 \leq y | X_1 = x_1) \\ &= P(x_1 + X_2 \leq y | X_1 = x_1) \\ &= P(X_2 \leq y - x_1) \\ &= \begin{cases} 0 & \text{if } y \leq x_1 \\ y-x_1 & \text{if } x_1 \leq y \leq x_1+1 \\ 1 & \text{if } y \geq x_1+1 \end{cases} \end{aligned}$$

$$f_{Y|X_1}(y|x_1) = \frac{dF_{Y|X_1}(y|x_1)}{dy} = \begin{cases} 1 & \text{if } x_1 \leq y \leq x_1+1 \\ 0 & \text{o.w.} \end{cases}$$

e. 6pts. Derive conditional CDF  $F_{X_1|Z}(x_1|z)$  and conditional PDF  $f_{X_1|Z}(x_1|z)$ .

$$f_{X_1|Z}(x_1|z) = \frac{f_{Z|X_1}(z|x_1)}{f_Z(z)} = \frac{f_{X_1}(x_1) \cdot f_{Z|X_1}(z|x_1)}{f_Z(z)} \text{ solve (f) first.}$$

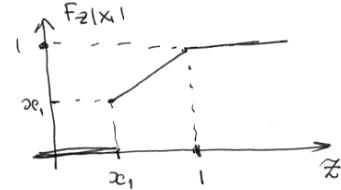
$$f_{X_1|Z}(x_1|z) = \frac{1}{2} \delta(x_1 - z) + \frac{1}{2z} \Pi(0, z) \quad z \in [0, 1] \\ 0 \leq x_1 \leq z$$

where  $\Pi(0, z) = \begin{cases} 1 & x_1 \in [0, z] \\ 0 & \text{o.w.} \end{cases}$

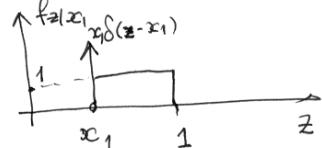
$$F_{X_1|Z}(x_1|z) = \int_{-\infty}^z f_{X_1|Z}(t|z) dt = \begin{cases} 0 & \text{if } x_1 < 0, \\ \frac{x_1}{2z} & 0 \leq x_1 < z \\ 1 & x_1 \geq z \end{cases}$$

f. 6pts. Derive conditional CDF  $F_{Z|X_1}(z|x_1)$  and conditional PDF  $f_{Z|X_1}(z|x_1)$ .

$$F_{Z|X_1}(z|x_1) = P(\max(X_1, X_2) \leq z | X_1 = x_1) \\ = P(X_1 \leq z \text{ and } X_2 \leq z | X_1 = x_1) \\ = \begin{cases} 0 & \text{if } x_1 > z \\ z & \text{if } x_1 \leq z \leq 1 \\ 1 & \text{if } z > 1 \end{cases}$$



$$f_{Z|X_1}(z|x_1) = \frac{dF_{Z|X_1}(z|x_1)}{dz} \\ = x_1 \cdot \delta(z - x_1) + \Pi(x_1, 1)$$



where  $\Pi(x_1, 1) = \begin{cases} 0 & z \leq x_1 \\ 1 & x_1 \leq z \leq 1 \\ 0 & z > 1 \end{cases}$

**Problem 2.3 (60 points) Communication through pulse positions.**

This problem asks you to analyze a communication system that communicates a message by sending a pulse during a set of time-slots associated with that message and just falls asleep (sends nothing) during other time slots. The receiver listens to all the time slots and has to decide which one corresponds to the transmitted message.

Explicitly:

- Let  $n$  be the number of samples corresponding to a single time slot.
- Let  $M$  represent the number of possible messages: each corresponding to a disjoint set of  $n$  samples.
- The receiver gets samples  $Y_{i,j}$  for  $i = 1, 2, \dots, M$  and  $j = 1, 2, \dots, n$ .

If message  $k$  was sent, then the  $Y_{k,j}$  are iid Bernoulli( $p$ ) with some specified  $p > \frac{1}{2}$ , while the rest of the  $Y_{l,j}$  are iid Bernoulli( $\frac{1}{2}$ ) for  $l \neq k$ .

The receiver operates by counting the number of 1's in each of the slots. Let  $Z_i$  be the number of 1's in the slot corresponding to the  $i$ -th message.

- a. 12pts What kind of random variable is  $Z_k$  if  $k$  is the transmitted message? What about  $Z_l$  for  $l \neq k$ ? Write out their PMFs, means, and variances.

$Z_k$  is Binomial with  $(n, p)$

$$P_{Z_k}(Z_k = k) = \binom{n}{k} p^k (1-p)^{n-k} \quad k=0, 1, \dots, n$$

$$E[Z_k] = np$$

$$\text{var}[Z_k] = np(1-p)$$

$Z_l$  is Binomial with  $(n, \frac{1}{2})$

$$P_{Z_l}(Z_l = l) = \binom{n}{l} \left(\frac{1}{2}\right)^n \quad l=0, 1, \dots, n$$

$$E[Z_l] = \frac{n}{2}$$

$$\text{var}[Z_l] = \frac{n}{4}$$

b. 10pts Suppose that the receiver decides to use a threshold  $T \in [0, 1]$  to decide if the message is present in slot  $i$ . If  $\frac{Z_i}{n} \geq T$ , we say the message is present. Otherwise we say the message is not present.

For a given threshold  $T$ , use Chebychev's inequality to give an upper bound on the probability of the true slot  $\frac{Z_k}{n}$  not crossing the threshold.

Repeat for the probability of a given false slot  $l \neq k$  crossing the threshold.

$$P\left(\frac{Z_k}{n} < T\right) = ? \quad \text{note } E\left[\frac{Z_k}{n}\right] = p \quad \text{var}\left[\frac{Z_k}{n}\right] = \frac{p(1-p)}{n}$$

Chebyshew inequality  $P(|X - \mu| \geq c) \leq \frac{\sigma^2}{c^2}$  where  $\mu = E[X]$   
 $\sigma^2 = \text{var}[X]$ .

$$\begin{aligned} * P\left(\frac{Z_k}{n} < T\right) &= P\left(\frac{Z_k}{n} - p < T - p\right) \\ &= P\left(\left(p - \frac{Z_k}{n}\right) > (p - T)\right) \\ &\leq P\left(\left|p - \frac{Z_k}{n}\right| > (p - T)\right) \end{aligned}$$

case 1.  $p - T > 0 ; T < p$

$$P\left(\frac{Z_k}{n} < T\right) \leq P\left(\left|\frac{Z_k}{n} - p\right| > p - T\right) \stackrel{\text{Chebyshew}}{\leq} \frac{p(1-p)}{n(p-T)^2}$$

case 2.  $p - T < 0 ; T > p$

$$P\left(\frac{Z_k}{n} < T\right) \leq 1$$

$$\begin{aligned} * P\left(\frac{Z_l}{n} \geq T\right) &= P\left(\frac{Z_l}{n} - \frac{1}{2} \geq T - \frac{1}{2}\right) \leq \\ &\leq P\left(\left|\frac{Z_l}{n} - \frac{1}{2}\right| \geq T - \frac{1}{2}\right) \leq \begin{cases} 1 & \text{if } T < \frac{1}{2} \\ \frac{1}{4n(T-\frac{1}{2})^2} & \text{if } T > \frac{1}{2} \end{cases} \end{aligned}$$

c. 12pts Repeat part [b] but use a Chernoff bounding argument (moment generating functions, etc.) instead of Chebychev's inequality.

$$M_{Z_K}(s) = (1-p + pe^s)^n$$

$$M_{Z_E}(s) = \left(\frac{1}{2} + \frac{1}{2}e^s\right)^n$$

$$\begin{aligned} P\left(\frac{Z_K}{n} < T\right) &\leq \min_{s<0} e^{-snT} M_{Z_K}(s) \\ &= \min_{s<0} e^{-snT} (1-p + pe^s)^n = \min_{s<0} \left[e^{-st}(1-p + pe^s)\right]^n \end{aligned}$$

$$\frac{d}{ds} \left[ \quad \right] = e^{-st}(-T) + pT e^{-st} - p(T-1) e^{-st} \cdot e^s = 0$$

$$e^s = \frac{T(1-p)}{p(1-T)} \Rightarrow s = \ln \frac{T(1-p)}{p(1-T)} \quad \begin{matrix} \text{substitute in} \\ \text{Chernoff bound} \\ \text{inequality.} \end{matrix}$$

$$P\left(\frac{Z_K}{n} < T\right) \leq \left[\left(\frac{p}{T}\right)^T \left(\frac{1-T}{1-p}\right)^{T-1}\right]^n$$

$$\begin{aligned} P\left(\frac{Z_E}{n} > T\right) &\leq \min_{s>0} e^{-snT} M_{Z_E}(s) \\ &= \min_{s>0} \left(\frac{1}{2}\right)^n \left[e^{-st}(1+e^s)\right]^n \end{aligned}$$

$$\begin{aligned} \frac{d}{ds} \left[ \quad \right] &= -Te^{-st} - (T-1)e^{-s(T-1)} = 0 \\ e^s &= \frac{T}{1-T} \Rightarrow s = \ln \frac{T}{1-T} \end{aligned}$$

$$\begin{aligned} P\left(\frac{Z_E}{n} > T\right) &\leq \left(\frac{1}{2}\right)^n \cdot \left(\frac{1-T}{T}\right)^{Tn} \cdot \left(\frac{1}{1-T}\right)^n \\ &\leq \left[\frac{1}{2} \cdot \left(\frac{1}{T}\right)^T \left(\frac{1}{1-T}\right)^{1-T}\right]^n \end{aligned}$$

d. 10pts Suppose now that  $n$  is quite large, and we want the probability of missing the true message to be less than 0.01. As a function of  $n$ , about how high can you put the threshold  $T$  while still meeting that constraint?

$$T = f(n) \text{ ? so that } P\left(\frac{Z_K}{n} < T\right) \leq 0.01$$

since  $n$  is quite large and  $Z_K = \sum_n$  Bernoulli

$$\frac{Z_K - np}{\sqrt{np(1-p)}} \sim N(0,1) \text{ according to CLT}$$

$$\begin{aligned} P\left(\frac{Z_K}{n} < T\right) &= P\left(\frac{Z_K - np}{\sqrt{np(1-p)}} < \frac{nT - np}{\sqrt{np(1-p)}}\right) \\ &= \Phi\left(\frac{nT - np}{\sqrt{np(1-p)}}\right) \leq 0.01 \end{aligned}$$

$$\frac{nT - np}{\sqrt{np(1-p)}} = \Phi^{-1}(0.01) = c \quad \text{note } \Phi^{-1}(\cdot) \text{ has two solutions}$$

$$T - p = \frac{1}{\sqrt{n}} \cdot c \cdot \sqrt{p(1-p)}$$

$$T = p + \frac{1}{\sqrt{n}} \cdot c \cdot \sqrt{p(1-p)} \cdot c$$

$$T = p + \frac{1}{\sqrt{n}} \underbrace{\Phi^{-1}(0.01)}_{-c, \text{ where } c>0} \cdot \sqrt{p(1-p)}$$

$$T = p - \frac{1}{\sqrt{n}} c \cdot \sqrt{p(1-p)}$$

we choose negative solution in the context of this problem.

e. 16pts Build on part c. Suppose now that the threshold  $T = p - \epsilon$  for some small  $\epsilon > 0$ . Let  $P_f$  be the probability that at least one of the false messages  $l \neq k$  has managed to cross the detection threshold  $T$ .

About how many messages  $M$  can we support while still maintaining  $P_f < 0.01$ ?

HINT: Use  $P(A \cup B) \leq P(A) + P(B)$  over and over again...

How does  $M$  scale with  $n$  for large  $n$ ?

$$T = p - \epsilon, \epsilon > 0.$$

Let  $A_i$  be the event that  $\frac{Z_i}{n} > T$  but  $i$  not sent.

$$P_f = P\left(\bigcup_{i=1}^{M-1} A_i\right) \leq \sum_{i=1}^{M-1} P(A_i) \text{ according to hint.}$$

$= (M-1) P(A_i)$  since  $A_i$ 's are events corresponding to  $Z_i$ 's that are identically dist.

$$P(A_i) = P\left(\frac{Z_i}{n} > T\right) \leq \left[\frac{1}{2} \left(\frac{1-T}{T}\right)^T \cdot \frac{1}{1-T}\right]^n \text{ from c).}$$

$$P_f \leq (M-1) P(A_i) \leq 0.01$$

$$\Rightarrow M \leq \frac{0.01}{P(A_i)} = 0.01 \cdot 2^n \cdot \left[\left(\frac{T}{1-T}\right)^T \cdot (1-T)\right]^n$$

$M$  scales exponentially with  $n$ .