Practice Midterm Solutions

Prob. 1.1

b)

The events A and B are independent if and only if $\mathbf{P}(A)\mathbf{P}(B) = \mathbf{P}(A \cap B) = \mathbf{P}(A)$, where the last equality follows from the fact that $A \subset B$. This can be the case if and only if $\mathbf{P}(A) = 0$ or $\mathbf{P}(B) = 1$.

c)

b. False. Consider X, Y both i.i.d. Bernoulli($\frac{1}{2}$). Let $Z = X \oplus Y$. The three are clearly not independent since Z is a deterministic function of X and Y. To see that they are pairwise independent, notice that X and Y are independent by assumption. It is clear that Z is Bernoulli($\frac{1}{2}$) on its own since $P(Z = 1) = P(X = 1, Y = 0) + P(X = 0, Y = 1) = \frac{1}{2}$. Z is independent from X because $P(Z = 1|X = 1) = P(Z = 0|X = 0) = P(Y = 0) = P(Y = 1) = P(Z = 1|X = 0) = P(Z = 0|X = 1)$. All these conditional probabilities are $\frac{1}{2}$ by the normalization property since $P(Z = 1|X = 1) + P(Z = 0|X = 1) = 1$ and similarly for conditioning on $X = 0$. Since conditioning on X leaves the probabilities for Z unchanged, Z is independent of X. As X and Y are symmetric relative to Z, the same argument also establishes that Z and Y are independent.

Prob. 1.2.

Problem 2: (46 points) Ramzi arrives first and always parks at the edge of a single row of n spaces and Danielle arrives later and always parks as close to Ramzi as possible.

(a) (16 pts) After Ramzi parks, there are $n-1$ consecutive spaces remaining. (We assume that $k < n-1$ which implies both that all k cars and Danielle will always find a spot.) The total number of ways k people can park in these $n-1$ remaining spots is $\binom{n-1}{k}$. Note that the closest open spot to Ramzi when Danielle arrives can be no further than k spots away, corresponding to the case that the k cars choose to fill the k closest spots to Ramzi i.e.,

$x \in \{0, 1, \ldots k\}$. If the closest open spot to Ramzi when Danielle arrives is x spots away, then x of the k cars must have parked in between Danielle and Ramzi and the remaining $k - x$ have *not* chosen the spot left open for Danielle. In other words, the number of ways the k cars can arrange themselves such that Danielle will end up x spots away from Ramzi is $\binom{n-2-x}{k-x}$. Hence, the PMF for random variable X is

$$p_X(x) = \frac{\binom{n-2-x}{k-x}}{\binom{n-1}{k}}, \quad x \in \{0, 1, \ldots k\} \quad .$$

(b) (15 pts) Let us consider the outcome of any day a "success" provided Danielle is able to park two spots or less from Ramzi, occurring with probability $p = \mathbf{P}(X \leq 2) = p_0 + p_1 + p_2$; otherwise, the outcome of any day is a "failure," occurring with probability $1 - p$. We are interested in the expected number of days *between two successes* of a sequence of independent Bernoulli trials; stated otherwise, given we just experienced a success, what is the expected number of trials between now and the next success. Thus, we are asking for the expected value of a geometric random variable Z with parameter p, describing the number of trials *up to and including the first success*, and subtracting one because we do not wish to include in our count the trial corresponding to the awaited success:

$$\mathbf{E}[Z-1] = \frac{1}{p} - 1 = \frac{1-p}{p} = \frac{1-p_0-p_1-p_2}{p_0+p_1+p_2} \quad .$$

(c) (15 pts) Again assuming independence on consecutive days, we have

$$\begin{aligned} \text{var}(Y) &= \text{var}\left(\frac{1}{m}\sum_{j=1}^{m} X_j\right) = \frac{1}{m^2}\left[\text{var}(X_1 + X_2 + \cdots + X_m)\right] \\ &= \frac{1}{m^2}\left[\text{var}(X_1) + \text{var}(X_2) + \cdots + \text{var}(X_m)\right] = \frac{1}{m^2}\left[m \cdot \text{var}(X)\right] = \frac{1}{m}\text{var}(X) \quad . \end{aligned}$$

In terms of the p_i's,

$$\text{var}(X) = \mathbf{E}[X^2] - (\mathbf{E}[X])^2 = \sum_{i=0}^{k} i^2 p_i - \left(\sum_{i=0}^{k} i p_i\right)^2$$

so

$$\text{var}(Y) = \frac{1}{m}\left(\sum_{i=0}^{k} i^2 p_i - \left(\sum_{i=0}^{k} i p_i\right)^2\right) \quad .$$

Prob. 1.3.

Problem 10. Let X and Y be independent exponential random variables with a common parameter λ.

(a) Find the transform associated with $aX + Y$, where a is a constant.

(b) Use the result of part (a) to find the PDF of $aX + Y$, for the case where a is positive and different than 1.

(c) Use the result of part (a) to find the PDF of $X - Y$.

Solution: (a) Let $Z = aX + Y$. We have

$$M_Z(s) = \mathbf{E}[e^{s(aX+Y)}] = \mathbf{E}[e^{saX}]\,\mathbf{E}[e^{s}Y] = M_X(sa)M_Y(s) = \frac{\lambda}{\lambda - sa} \cdot \frac{\lambda}{\lambda - s},$$

for $s < \lambda$.

(b) We will express the transform of Z in the form

$$M_Z(s) = \frac{c}{\lambda - sa} + \frac{d}{\lambda - s}.$$

We have

$$c = M_Z(s)(\lambda - sa)\Big|_{s=\lambda/a} = \frac{\lambda^2}{\lambda - \lambda/a} = \frac{a\lambda}{a-1},$$

and

$$d = M_Z(s)(\lambda - s)\Big|_{s=\lambda} = \frac{\lambda^2}{\lambda - a\lambda} = -\frac{\lambda}{a-1}.$$

Thus,

$$M_Z(s) = \frac{\lambda}{a-1}\left(\frac{a}{\lambda - sa} - \frac{1}{\lambda - s}\right) = \frac{a}{a-1}\cdot\frac{\lambda/a}{\lambda/a - s} - \frac{1}{a-1}\cdot\frac{\lambda}{\lambda - s}.$$

We recognize this as the transform associated with the PDF

$$f_X(x) = \frac{a}{a-1}e^{-\lambda x/a} - \frac{1}{a-1}e^{-\lambda x}, \qquad x \geq 0.$$

(c) From part (a) we have that the transform of $-Y$ is equal to $\lambda/(\lambda+s)$. With $Z = X - Y$, we have

$$M_Z(s) = \frac{\lambda^2}{(\lambda - s)(\lambda + s)} = \frac{1}{2}\left(\frac{\lambda}{\lambda - s} + \frac{\lambda}{\lambda + s}\right),$$

which we recognize as the transform of a mixture of two random variables, one distributed as X, the other distributed as $-Y$. It follows that

$$p_Z(z) = \begin{cases} e^{-\lambda z}/2, & \text{if } z \geq 0, \\ e^{\lambda z}/2, & \text{if } z \leq 0, \end{cases}$$

or

$$p_Z(z) = \frac{1}{2}e^{-\lambda|z|}.$$