

SPRING 2005 EE126 PRACTICE MIDTERM 2

PROBLEM 2.1.

a. TRUE

$$\begin{aligned}\text{var}\left(\frac{X+Y}{2}\right) &= \frac{1}{4} (\text{var}(X) + \text{var}(Y) + 2\text{cov}(X, Y)) \\ &\leq \frac{1}{4} (2 \max\{\text{var}(X), \text{var}(Y)\} + 2\text{cov}(X, Y))\end{aligned}$$

$$\begin{aligned}\text{since } \beta \leq 1 \quad \text{cov}(X, Y) &\leq \sqrt{\text{var}(X)\text{var}(Y)} \\ &\leq \max\{\text{var}(X), \text{var}(Y)\}\end{aligned}$$

Therefore,

$$\begin{aligned}\text{var}\left(\frac{X+Y}{2}\right) &\leq \frac{1}{4} [2 \max\{\text{var}(X), \text{var}(Y)\} + 2 \max\{\text{var}(X), \text{var}(Y)\}] \\ &= \max\{\text{var}(X), \text{var}(Y)\}\end{aligned}$$

b. FALSE

Counterexample 1

$$\text{Let } X_1 = -X_2, X_3 = -X_4, \dots, X_{n-1} = X_n$$

$$\text{Then } Y = \frac{\sum X_i}{\sqrt{n}} = 0$$

Counterexample 2

$$X_1 = X_2 = \dots = X_n \quad X = \sum X_i \sim N(0, n^2)$$

$$Y = \frac{\sum X_i}{\sqrt{n}} \sim N(0, n)$$

c. FALSE

For the moment generating function to be well defined, the tail probabilities of the r.v. need to die faster than some exponential. This needs to be true for general r.v.

Consider X with density

$$f_X(x) = \begin{cases} 0 & \text{if } x < 1 \\ \frac{1}{x^2} & \text{if } x \geq 1 \end{cases}$$

This is a proper density since

$$\int_{-\infty}^{+\infty} f_X(x) dx = \int_1^{+\infty} \frac{1}{x^2} dx = 1$$

But the moment generating function is not defined for any value $s > 0$

since

$$M_X(s) = \int_1^{+\infty} \frac{e^{sx}}{x^2} dx = \infty$$

because $e^s > 1$ and $\lim_{x \rightarrow \infty} \frac{(e^s)^x}{x^2} = \infty$

PROBLEM 2.2.

X, Y, Z_1, Z_2 independent rv. known means and variances.

$$U = X - Y + Z_1$$

$$V = X + Y + Z_2$$

$$\hat{X} = \text{LLSE}(X|U, V) = ? \quad \text{m.m.s.e.} \quad E[(X - \hat{X})^2] = ?$$

$$\hat{Y} = \text{LLSE}(Y|U, V) = ? \quad \text{m.m.s.e.} \quad E[(Y - \hat{Y})^2] = ?$$

Note that

$$U_1 = U + V = 2X + Z_1 + Z_2$$

$$V_1 = V - U = 2Y + Z_2 - Z_1$$

$$E(U_1) = 2m_X + m_{Z_1} + m_{Z_2}$$

$$\text{var}(U_1) = 4\sigma_X^2 + \sigma_{Z_1}^2 + \sigma_{Z_2}^2$$

$$E(V_1) = 2m_Y + m_{Z_2} - m_{Z_1}$$

$$\text{var}(V_1) = 4\sigma_Y^2 + \sigma_{Z_1}^2 + \sigma_{Z_2}^2$$

$$E(U_1 V_1) = E[(2X + Z_1 + Z_2)(2Y + Z_2 - Z_1)]$$

$$= E[Z_2^2] - E[Z_1^2]$$

Note that when $m_{Z_1} = m_{Z_2}$ and $\text{var}(Z_1) = \text{var}(Z_2)$ then U_1 and V_1 are uncorrelated.

$$\hat{X} = aU_1 + bV_1 + c$$

$$\hat{Y} = dU_1 + eV_1 + f$$

$E[(X - aU_1 - bV_1 - c)^2]$ is minimized for

$$\frac{\partial}{\partial a} [] = 0 \Rightarrow E[(X - aU_1 - bV_1 - c) \cdot U_1] = 0$$

$$\frac{\partial}{\partial b} [] = 0 \Rightarrow E[(X - aU_1 - bV_1 - c) \cdot V_1] = 0$$

$$\frac{\partial}{\partial c} [] = 0 \Rightarrow E[(X - aU_1 - bV_1 - c)] = 0.$$

$$\begin{bmatrix} E(U_1^2) & E(U_1V_1) & E(U_1) \\ E(U_1V_1) & E(V_1^2) & E(V_1) \\ E(U_1) & E(V_1) & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} E(XU_1) \\ E(XV_1) \\ E[X] \end{bmatrix}$$

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} E(U_1^2) & E(U_1V_1) & E(U_1) \\ E(U_1V_1) & E(V_1^2) & E(V_1) \\ E(U_1) & E(V_1) & 1 \end{bmatrix}^{-1} \begin{bmatrix} E[XU_1] \\ E[XV_1] \\ E[X] \end{bmatrix}$$

similarly

$$\begin{bmatrix} d \\ e \\ f \end{bmatrix} = \begin{bmatrix} E(U_1^2) & E(U_1V_1) & E(U_1) \\ E(U_1V_1) & E(V_1^2) & E(V_1) \\ E(U_1) & E(V_1) & 1 \end{bmatrix}^{-1} \begin{bmatrix} E(YU_1) \\ E(YV_1) \\ E[Y] \end{bmatrix}$$

b. $X \sim \text{Bernoulli}(P)$

$$N_0 \sim N(0, \sigma_0^2)$$

$$N_1 \sim N(0, \sigma_1^2)$$

$$N_2 \sim N(0, \sigma_2^2)$$

$$Y_1 = X + N_0 + N_1$$

$$Y_2 = X + N_0 + N_2$$

$\hat{X} = \text{LLSE}(X|Y_1, Y_2) = aY_1 + bY_2 + c$ is one way of solving it.

but note that $X + N_0 = Z$ is a common term,

that can be estimated first.

$$Y_1 = Z + N_1$$

$$Y_2 = Z + N_2$$

$$\hat{Z} = a_1 Y_1 + a_2 Y_2 + a_3$$

Coeff. a_1, a_2, a_3 can be obtained from

$$\begin{bmatrix} E[Y_1^2] & E[Y_1 Y_2] & E[Y_1] \\ E[Y_1 Y_2] & E[Y_2^2] & E[Y_2] \\ E[Y_1] & E[Y_2] & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} E[Z Y_1] \\ E[Z Y_2] \\ E[Z] \end{bmatrix}$$

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} p + \sigma_o^2 + \sigma_1^2 & p + \sigma_o^2 & p \\ p + \sigma_o^2 & p + \sigma_o^2 + \sigma_2^2 & p \\ p & p & 1 \end{bmatrix}^{-1} \begin{bmatrix} p \\ p \\ p \end{bmatrix}$$

and then

$$\hat{X} = b \hat{Z} + c$$

where b, c can be found as:

$$\begin{bmatrix} E[\hat{Z}^2] & E[\hat{Z}] \\ E[\hat{Z}] & 1 \end{bmatrix} \begin{bmatrix} b \\ c \end{bmatrix} = \begin{bmatrix} E[X\hat{Z}] \\ E[X] \end{bmatrix}$$

$$E[\hat{Z}] = a_1 p + a_2 p + a_3$$

$$E[X\hat{Z}] = a_1 p + a_2 p + a_3 p$$

$$E[\hat{Z}\hat{Z}] = a_1^2(p + \sigma_o^2 + \sigma_1^2) + a_2^2(p + \sigma_o^2 + \sigma_2^2) + a_3^2 + a_1 a_2 (p + \sigma_o^2) + a_1 a_3 p + a_2 a_3 p$$

So, the $\hat{X} = b(a_1 Y_1 + a_2 Y_2 + a_3) + c$.

$$\text{m.m.s.e. } E[(X - \hat{X}) \cdot X]$$

| Spring 2005 EE126 Practice Midterm 2

Problem 2.3

$$X_i \rightarrow \boxed{\text{channel}} \rightarrow Y_i = X_i \cdot Z_i$$

$$P_{X_i}(x_i) = \begin{cases} \frac{1}{2} & x_i = 1 \\ \frac{1}{2} & x_i = -1 \\ 0 & \text{otherwise} \end{cases}$$

$$P_{Z_i}(z_i) = \begin{cases} \delta & z_i = 0 \\ 1-\delta & z_i = 1 \\ 0 & \text{otherwise} \end{cases}$$

a. $P(\tilde{X}_i^n = \check{X}_i^n) = P\left(\bigcap_{i=1}^n \{\tilde{X}_i = \check{X}_i\}\right)$

by independence $\Rightarrow P\left(\bigcap_{i=1}^n \{\tilde{X}_i = \check{X}_i\}\right) = \prod_{i=1}^n P(\tilde{X}_i = \check{X}_i)$

$$\begin{aligned} \text{For any } i \leq n, \quad P(\tilde{X}_i = \check{X}_i) &= P(\tilde{X}_i = \check{X}_i \mid \check{X}_i = 1)P(\check{X}_i = 1) + \\ &\quad P(\tilde{X}_i = \check{X}_i \mid \check{X}_i = -1)P(\check{X}_i = -1) \\ &= P(\tilde{X}_i = 1 \mid \check{X}_i = 1) \frac{1}{2} + P(\tilde{X}_i = -1 \mid \check{X}_i = -1) \frac{1}{2} \\ \text{by independence} &= P(\tilde{X}_i = 1) \frac{1}{2} + P(\tilde{X}_i = -1) \frac{1}{2} \\ &= \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \frac{1}{2} \\ &= \frac{1}{2} \end{aligned}$$

$$\therefore \prod_{i=1}^n P(\tilde{X}_i = \check{X}_i) = \left(\frac{1}{2}\right)^n$$

b. By the channel model, for each $1 \leq i \leq n$

$$Y_i = \tilde{X}_i Z_i$$

$$\text{Case 1: } Z_i = 1, \quad Y_i = \tilde{X}_i$$

$$\text{Case 2: } Z_i = 0, \quad Y_i = 0$$

Since in either case Y_i is compatible with \tilde{X}_i
 $\therefore Y_i$ is compatible with \tilde{X}_i for all i
 i.e. Y_i^n is always compatible with \tilde{X}_i^n

c. Let \tilde{X}_i^n be the true codeword that produces Y_i^n

Let \check{X}_i^n be any other codeword.

$$\tilde{X}_i^n \perp \check{X}_i^n \text{ (by random generation)}$$

$$P\left(\bigcap_{i=1}^n Y_i = \check{X}_i \cdot Z_i\right) = \prod_{i=1}^n P(Y_i = \check{X}_i \cdot Z_i) \text{ by indep.}$$

2

$$\begin{aligned}
 & \text{For each } 1 \leq i \leq n \quad P(Y_i = \check{x}_i \cdot z_i) \\
 &= P(Y_i = \check{x}_i \cdot z_i | z_i=1)P(z_i=1) + \\
 &\quad P(Y_i = \check{x}_i \cdot z_i | z_i=0)P(z_i=0) \\
 &= P(Y_i = \check{x}_i | z_i=1)(1-\delta) + P(Y_i = 0 | z_i=0)\delta \\
 &= P(\check{x}_i = \check{x}_i | z_i=1)(1-\delta) + P(\check{x}_i \cdot z_i = 0 | z_i=0)\delta \\
 &= P(\check{x}_i = \check{x}_i)(1-\delta) + P(0=0 | z_i=0)\delta \\
 &\quad \uparrow \text{by indep} \\
 &= \frac{1}{2}(1-\delta) + \delta = \frac{1}{2} + \frac{1}{2}\delta \\
 &\therefore P\left(\bigwedge_{i=1}^n Y_i = \check{x}_i \cdot z_i\right) = \left(\frac{1}{2} + \frac{1}{2}\delta\right)^n
 \end{aligned}$$

d. For $1 \leq k \leq M$, let $x_i^n(k)$ denote the k th codeword.

Let A_i be the event $\{Y_i^n \text{ is compatible with } x_i^n(i)\}$

$$\begin{aligned}
 P_e &= P(Y_i^n \text{ is compatible with at least 2 codewords}) \\
 &= \sum_{j=1}^M P(Y_i^n \text{ is compatible w/ at least 2 codewords} \mid x_i^n(j) \text{ is the true one}) \\
 &\quad P(x_i^n(j) \text{ is the true one}) \\
 &= \sum_{j=1}^M P(Y_i^n \text{ is compatible w/ at least 1 other codeword} \mid x_i^n(j) \text{ is true}) \\
 &\quad P(x_i^n(j) \text{ is true}) \\
 &= \sum_{j=1}^M P\left(\bigvee_{\substack{i \neq j \\ 1 \leq i \leq M}} A_i \mid x_i^n(j) \text{ is true}\right) P(x_i^n(j) \text{ is true})
 \end{aligned}$$

Union bound $P(A \cup B) \leq P(A) + P(B) \Rightarrow P\left(\bigcup_{i=1}^n A_i\right) \leq \sum_{i=1}^n P(A_i)$

$$\begin{aligned}
 &\leq \sum_{j=1}^M \sum_{\substack{i \neq j \\ 1 \leq i \leq M}} P(A_i \mid x_i^n(j) \text{ is true}) P(x_i^n(j) \text{ is true}) \\
 &= \sum_{j=1}^M (M-1) \left(\frac{1}{2} + \frac{1}{2}\delta\right)^n P(x_i^n(j) \text{ is true}) \\
 &= (M-1) \left(\frac{1}{2} + \frac{1}{2}\delta\right)^n \underbrace{\sum_{j=1}^M P(x_i^n(j) \text{ is true})}_{\text{This is 1 since one of the codewords is true}} \\
 &= (M-1) \left(\frac{1}{2} + \frac{1}{2}\delta\right)^n \\
 &\leq M \left(\frac{1}{2} + \frac{1}{2}\delta\right)^n
 \end{aligned}$$

3

$$e. \quad P_e \leq M \left(\frac{1}{2} + \frac{1}{2}\delta \right)^n$$

$M \left(\frac{1}{2} + \frac{1}{2}\delta \right)^n < \varepsilon$ will guarantee $P_e < \varepsilon$

$$\therefore M < \varepsilon \left(\frac{1}{2} + \frac{1}{2}\delta \right)^{-n}$$

In terms of rate, $\lg M$

$$\lg M < \lg \varepsilon - n \lg \left(\frac{1+\delta}{2} \right)$$

$$\lg M < n \lg \left(\frac{2}{1+\delta} \right) - \lg \frac{1}{\varepsilon} \quad \text{linear in } n.$$

f. Condition on $L = l$

$$P(Y_1^n = X_1^n \cdot Z_1^n \mid L=l)$$

= $P(\text{The } (n-l) \text{ non-erased bits of } Y_1^n \text{ equal to } X_1^n)$

$$= \left(\frac{1}{2} \right)^{n-l}$$

$$P_{e|L=l} \leq M \left(\frac{1}{2} \right)^{n-l}$$

$$g. \quad P(L \geq n(\delta + \beta)) \leq \min_{s \geq 0} e^{-sn(\delta+\beta)} M_L(s) \text{ by Chernoff bound}$$

$$M_L(s) = (1 - \delta + s e^s)^n \quad \text{since } L \text{ is Binomial}(n, \delta)$$

~~$$e^{-sn(\delta+\beta)} M_L(s) = e^{-sn(\delta+\beta)} ((1-\delta) + s e^s)^n$$~~

$$= [e^{-s(\delta+\beta)} ((1-\delta) + s e^s)]^n$$

$$= [(1-\delta) e^{-s(\delta+\beta)} + s e^{s(1-\delta-\beta)}]^n$$

To minimize the above, we take the derivative w/rsp to s.

$$-(1-\delta)(\delta+\beta) e^{-s(\delta+\beta)} + \delta(1-\delta-\beta) e^{s(1-\delta-\beta)} = 0$$

$$e^{s(1-\delta-\beta)} = \frac{(1-\delta)(\delta+\beta)}{\delta(1-\delta-\beta)}$$

$$e^s = \frac{(1-\delta)(\delta+\beta)}{\delta(1-\delta-\beta)} \quad s = \ln \frac{(1-\delta)(\delta+\beta)}{\delta(1-\delta-\beta)} > 0$$

4

$$\begin{aligned}
 P(L \geq n(\delta + \beta)) &\leq \left[(1-\delta) \left(\frac{\delta(1-\delta-\beta)}{(1-\delta)(\delta+\beta)} \right)^{\delta+\beta} + \delta \left(\frac{(1-\delta)(\delta+\beta)}{\delta(1-\delta-\beta)} \right)^{1-\delta-\beta} \right]^n \\
 &= \left[\left(\frac{\delta(1-\delta-\beta)}{(1-\delta)(\delta+\beta)} \right)^{\delta+\beta} \left(1-\delta + \delta \frac{(1-\delta)(\delta+\beta)}{\delta(1-\delta-\beta)} \right) \right]^n \\
 &= \left[\left(\frac{\delta(1-\delta-\beta)}{(1-\delta)(\delta+\beta)} \right)^{\delta+\beta} \left(\frac{(1-\delta)(1-\delta-\beta) + (1-\delta)(\delta+\beta)}{1-\delta-\beta} \right) \right]^n \\
 &= \left[\left(\frac{\delta(1-\delta-\beta)}{(1-\delta)(\delta+\beta)} \right)^{\delta+\beta-1} \frac{\delta}{\delta+\beta} \right]^n
 \end{aligned}$$

Since $\delta < \delta + \beta$, $1-\delta-\beta < 1-\delta$

$$\therefore \left(\frac{\delta(1-\delta-\beta)}{(1-\delta)(\delta+\beta)} \right)^{\delta+\beta-1} \frac{\delta}{\delta+\beta} < 1$$

$\therefore P(L \geq n(\delta + \beta)) \rightarrow 0$ exponentially fast.

$$\begin{aligned}
 h. P_e &= \sum_{\ell=1}^n P_{e|L=\ell} P(L=\ell) \\
 &= \sum_{\ell=1}^{\lfloor n(\delta+\beta) \rfloor} P_{e|L=\ell} P(L=\ell) + \sum_{\ell=\lfloor n(\delta+\beta) \rfloor + 1}^n P_{e|L=\ell} P(L=\ell) \\
 &\leq \sum_{\ell=1}^{\lfloor n(\delta+\beta) \rfloor} M \left(\frac{1}{2} \right)^{n-\ell} \cdot 1 + \sum_{\ell=\lfloor n(\delta+\beta) \rfloor + 1}^n 1 \cdot C^n \\
 &= M \left(\frac{1}{2} \right)^n \sum_{\ell=1}^{\lfloor n(\delta+\beta) \rfloor} \left(\frac{1}{2} \right)^{-\ell} + (n - \lfloor n(\delta+\beta) \rfloor) C^n \quad \text{where } C^n \text{ is the Chernoff bound} \\
 &= M \left(\frac{1}{2} \right)^n \left(2^{\lfloor n(\delta+\beta) \rfloor + 1} - 2 \right) + (n - \lfloor n(\delta+\beta) \rfloor) C^n
 \end{aligned}$$