Spring 2005: EECS126 Take Home Pseudo-Final
Assigned: May 5 in class
Due: May 10 in class

- Aside from your pen/pencil, you are allowed to use only your textbook and blank sheets of paper. **Collaboration is strictly forbidden** and you are not alllowed to use any notes, calculating devices, or the Internet.
- You are obliged by the honor system to spend no longer than 4 hours from the time you first open the exam to the time that you finish working on it. (Eat something before, go to the bathroom, turn off your cellphone/pager and stay away from your computer so you are not interrupted and thereby lose time.) Mark the start and stop time on your exam.
- If you wish to spend longer than that, you must clearly mark the sections that you have worked on once the time limit is up. You will receive some partial credit for that extra work.
- This exam is optional. You do not have to do it. But if you do well on it, it will mitigate your scores from the first two midterms.

Problem 3.1 *True or False. Prove or show a counterexample:*

a. *12pts. If a random variable has finite second moment, then it has finite first moment.*

b. *12pts. Let A_1, A_2, A_3 be events with $0 < P(A_3) < 1$. Suppose A_1 and A_2 are conditionally independent given A_3 and are also conditionally independent given A_3^c. Then A_1 and A_2 are independent.*

c. 12pts. For any random variable X and any $a > 0$ we have

$$P(|X| < a) \leq a^2 E[\frac{1}{X^2}]$$

Problem 3.2 *"Pairing up"*

In a particularly simplified society, there are exactly n young males and exactly n young females. In their mythology, for every male i there is exactly 1 female who is destined to be his soulmate and amazingly he is also destined to be her soulmate.[1]

a. 15 pts. *In one case, the strategy for pairing up individuals is as follows. At every time, two currently unattached males are randomly and uniformly chosen and paired with a randomly and uniformly chosen single unattached female. If either pairing turns out to be soulmates, then that pair drops out of the pool of unattached individuals, while the remaining male returns to the pool. Otherwise, all three enjoy themselves and all return to the pool. The process starts with all $2n$ individuals in the single pool and it will terminate when everyone is married.*

Write out a model for this and

1. find the probability that the first randomly drawn group will result in finding a pair of soulmates.

2. give an exact expression for the expected time till everyone is married.

How does this time scale with increasing n?

HINT: Think about setting this up as a Markov chain with n states, 1 absorbing ...

[1] As you can see, this model has almost no connection to reality as in the real world there are a host of good pairing possibilities for any individual.

Extra Space:

b. *20 pts. In another case, the society decides to randomly deploy matchmakers to help get people paired with their soulmates. For every* $2n$ *singles, there are only about* $\lceil K \ln n \rceil$ *match-making*[2] *elders. Every single person* i *has an independent probability* p *of knowing any particular matchmaker* k*. If any matchmaker knows both sides of a pair of soulmates, then that pair will be brought together and married. A pair cannot be brought together if they know no matchmakers in common. If there is any pair of soulmates that can not be brought together, it is considered a tragedy for the entire society.*

1. What is the probability that the pair (i, j) *do not have matchmaker* k *in common?*

2. Give an expression for the probability that a specified pair of soulmates (i, j) *will not be brought together.*

3. What is the best bound you can give on the probability of a tragedy occurring?

4. As a function of the parameter K*, how does the probability of tragedy change in the limit of large* n*?*

HINT: Draw a picture to help you see what is going on. In the limit of large n feel free to just use $K \ln n$ instead of $\lceil K \ln n \rceil$. For studying the limit of large n, looking at ln of the probability is easier than studying the probability itself.

[2] $\lceil x \rceil$ refers to the integer you get by rounding x upward.

Extra space:

c. *20 pts. After a pair of supposed soulmates gets married, they find themselves in a Markov process with three states:*

Honeymoon: *From this state, they continue in this state for the next period with probability $\frac{3}{4}$ but otherwise fall into the "fighting state."*

Fighting: *From this state, they continue fighting in the next period with probability $\frac{1}{2}$ but otherwise make up and enter the "Civil" state.*

Civil: *From this state, they stay civil in the next period with probability $\frac{2}{3}$ but otherwise fall into the "fighting state."*

They start out in the "honeymoon state" once they get married.

1. Which states are recurrent and which are transient? How many recurrent classes are there?

2. For arbitrary time t after they are married, give an expression for the probability that they are currently fighting.

3. As t gets very large, what does the probability of fighting tend toward?

Extra workspace:

Problem 3.3 *"Transmitters With Messages"*

Transmitters A and B independently send messages to a single receiver in a Poisson manner, with rates of λ_A and λ_B respectively. All messages are so brief that we may safely assume that they occupy single points in time. The number of words in the i-th message, regardless of the source that is transmitting it, is a random variable W_i with PMF:

$$P_W(w) = \begin{cases} \frac{1}{2} & \textit{if } w = 1 \\ \frac{1}{3} & \textit{if } w = 2 \\ \frac{1}{6} & \textit{if } w = 3 \\ 0 & \textit{otherwise} \end{cases}$$

and is independent of anything else.

a. *5 pts. In an interval of duration t, what is the probability that exactly 9 messages are received?*

b. *5 pts. Let N be the total number of words received in an interval of duration t. What is $E[N]$?*

c. 10 pts. Determine the PDF of the length of time from 0 *until the receiver receives the third message that is exactly 2 words long from transmitter A.*

d. 10 pts. What is the probability that exactly 2 *out of the next* 5 *messages received will be from transmitter A.*