Problem 6.1

Solution: It is useful to think of $X$ as a uniform random variable in $[\frac{\pi}{2}; \frac{3\pi}{2}]$, we have

$$F_X(x) = \begin{cases} 
0, & x \leq \frac{\pi}{2} \\
\frac{x}{\frac{3\pi}{2}}, & \frac{\pi}{2} < x \leq \frac{3\pi}{2} \\
1, & x \geq \frac{3\pi}{2}
\end{cases}$$

For a particular value $y \in [-1;1]$ of the random variable $Y$, we have two values $x_1, x_2$ of the random variable $X$ corresponding to it. Requiring $P(\sin(X) \leq y)$ is equivalent to requiring $P(X \leq x_1) + P(X \geq x_2)$. By symmetry of the $\sin(y)$ function, $P(X \leq x_1) = P(X \geq x_2)$. See Figure 1.

Thus,

$$P(Y \leq y) = P(\sin(X) \leq y) = 2P(X \leq \sin^{-1}(y)) = 2 \sin^{-1}(y)$$

for $y \in [-1;1]$. From which,

$$f_Y(y) = \frac{d}{dy} P(Y \leq y) = \frac{d}{dy} 2 \sin^{-1}(y) = \frac{1}{\pi \sqrt{1 - y^2}}$$

for $y \in [-1;1]$

Problem 6.2

Solution:

Let $G$ denote the event that Dino has a good day and let $B$ denote the event that Dino has a bad day. We are given that $P(G) = P(B) = 0.5$. Let $T$ be the time it takes Dino to cook a souffle, so that $f_{T | G}$ is uniform between $1/2$ and 1, and $f_{T | B}$ is uniform between $1/2$ and $3/2$. We need to find $P(B \mid T \leq 3/4)$.

Using Bayes's rule, we have

$$P(B \mid T \leq 3/4) = \frac{P(T \leq 3/4 \mid B)P(B)}{P(T \leq 3/4)} = \frac{P(T \leq 3/4 \mid B)P(B)}{P(T \leq 3/4 \mid B)P(B) + P(T \leq 3/4 \mid G)P(G)}$$

Evaluating this expression, we find that

$$P(B \mid T \leq 3/4) = \frac{\left(\frac{1}{4}\right)\left(\frac{1}{2}\right)}{\left(\frac{1}{4}\right)\left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)} = \frac{1}{3}.$$
\[ \sin(x) \leq y \Rightarrow x \leq x_1 \text{ and } x \geq x_2 \]
Problem 6.3
(a) Let $S$ and $R$ be the events that the day is sunny and rainy, respectively. We are given that $f_{X|S}$ and $f_{X|R}$ are uniform PDFs over the intervals $[30, 45]$ and $[40, 60]$, respectively. Moreover we know that $P(S) = 2/3$ and $P(R) = 1/3$. If follows that

$$f_X(x) = P(S)f_{X|S}(x) + P(R)f_{X|R}(x)$$

$$= \begin{cases} 
(2/3)(1/15) = 0.0444, & \text{if } 30 \leq x < 40, \\
(2/3)(1/15) + (1/3)(1/20) = 0.0611, & \text{if } 40 \leq x \leq 45, \\
(1/3)(1/20) = 0.0167, & \text{if } 45 < x \leq 60.
\end{cases}$$

As for the expected value and the variance of $X$, we have

$$E[X] = E[X \mid S]P(S) + E[X \mid R]P(R) = \frac{30 + 45}{2} \cdot \frac{2}{3} + \frac{40 + 60}{2} \cdot \frac{1}{3} = \frac{125}{3} = 41.67,$$

$$E[X^2] = E[X^2 \mid S]P(S) + E[X^2 \mid R]P(R)$$

$$= \frac{2}{3} \cdot \frac{1}{15} \int_{30}^{45} x^2 \, dx + \frac{1}{3} \cdot \frac{1}{20} \int_{40}^{60} x^2 \, dx$$

$$= 1794.4,$$

and

$$\text{var}(X) = 1794.4 - (41.67)^2 = 58.01.$$

(b) $P(R \mid X = 45) = \frac{P(R)f_X(45 \mid R)}{f_X(45)} = \frac{(1/3)(1/20)}{0.0611} = 0.2733.$$

(c) Let $V$ be the average speed. We have $V = 20/X$, so that

$$f_V(v) = f_X \left( \frac{20}{v} \right) \left| \frac{d}{dv} \left( \frac{20}{v} \right) \right|$$

$$= \frac{20}{v^2} f_X \left( \frac{20}{v} \right)$$

$$= \begin{cases} 
1/(3v^2), & \text{if } 20/60 \leq v < 20/45, \\
55/(45v^2), & \text{if } 20/45 \leq v \leq 20/40, \\
8/(9v^2), & \text{if } 20/40 < v \leq 20/30,
\end{cases}$$

$$E[V] = \int_{20/60}^{20/45} \frac{v}{2v^2} \, dv + \int_{20/45}^{20/40} \frac{25v}{18v^2} \, dv + \int_{20/40}^{20/30} \frac{8v}{9v^2} \, dv = 0.496,$$

$$E[V^2] = \int_{20/60}^{20/45} \frac{v^2}{2v^2} \, dv + \int_{20/45}^{20/40} \frac{25v^2}{18v^2} \, dv + \int_{20/40}^{20/30} \frac{8v^2}{9v^2} \, dv = 0.253,$$

and

$$\text{var}(V) = 0.253 - (0.496)^2 = 6.984 \cdot 10^{-3}.$$
Problem 6.4

(a) We have

\[ P(B \mid A) = \frac{P(A \cap B)}{P(A)} = \frac{P(X < Y \leq 0.5)}{1 - P(Y \geq 0.5)} = \frac{0.25}{1 - 0.25} = \frac{1}{3}. \]

(b) We have

\[ f_{X \mid Y}(x \mid 0.5) = \frac{f_{X,Y}(x,0.5)}{f_Y(0.5)} = \begin{cases} 2, & \text{if } 0 < x \leq 0.5, \\ 0, & \text{otherwise}. \end{cases} \]

The conditional expectation is 1/4 and the conditional variance is \((0.5)^2/12 = 1/3\).

(c) We have

\[ f_{X,Y \mid B}(x,y) = \frac{f_{X,Y}(x,y)P(B)}{P(B)}, \quad \text{if } (x,y) \in B, \]
\[ = \begin{cases} 4, & \text{if } (x,y) \in B, \\ 0, & \text{otherwise}. \end{cases} \]

Therefore,

\[ f_{X \mid B}(x) = \int_0^1 f_{X,Y \mid B}(x,y) \, dy \]
\[ = \begin{cases} \int_x^{1-x} 4 \, dy = 4(1-2x), & \text{if } 0 < x \leq 0.5, \\ 0, & \text{otherwise}. \end{cases} \]

(d) We have

\[ E[XY] = \int_0^1 \int_0^{1-x} 2xy \, dy \, dx = \int_0^1 x(1-x)^2 \, dx = \frac{1}{12}. \]

(e) Let \( Z = Y/X \). If \( z \geq 0 \),

\[ f_Z(z) = \frac{d}{dz} P \left( \frac{Y}{X} \leq z \right) = \frac{d}{dz} \left( \frac{z}{z+1} \right) = \frac{1}{(z+1)^2}. \]