Problem 9.1

(a) A continuous random variable $U$ has a pdf of the form:

$$f_U(u) = ae^{bu^2+cu}, \quad -\infty < u < +\infty$$

where $a, b, c$ are constants. $a$ and $b$ are non-zero.

i) Can you say anything definitive about the signs of $a$, $b$ and $c$?

ii) Does $U$ have to be Gaussian? If so, express its mean and variance in terms of $a$, $b$ and $c$ in the simplest way. If not, give an example of $U$ with pdf of the above form but is not Gaussian.

(b) Let $X \sim N(0; \sigma^2)$, find the moment generating function of $X$, $M_X(s)$.

(c) Let $X \sim N(0; \sigma^2)$, $Z \sim N(0; \sigma^2)$ and we have a noisy observation $Y = X + Z$. The RVs $X$ and $Z$ are independent. Find from first principles the MMSE estimate of $X$ given $Y$ and the resulting minimum mean square error. (Hint: you may find your answer to part (a)(ii) useful in simplifying the calculations.).

Solution: See the handwritten solutions in the previous discussion section.

Problem 9.2

Let $X$ be a Bernoulli random variable with parameter $p$. A random variable $Z$ is a mixture of Gaussian random variables and is defined as

$$Z = XY_1 + (1 - X)Y_2$$

where $Y_1$ is $N(-1, \sigma^2)$ and $Y_2$ is $N(1, \sigma^2)$.

We have access to a noisy observation of $Z$, which we denote as $V$. $V$ is given by $Z$ corrupted by additive gaussian noise, i.e.

$$V = Z + W$$

where $\text{Var}(W) = \sigma_W^2$. Given $V$, find the estimator of $Z$ that minimizes the mean squared error.

Solution: The estimator that minimizes the mean squared error is $E[Z|V]$. The expectation can be computed by further conditioning over $X$:

$$E[Z|V] = E[Z|V, X = 1]P(X = 1|V) + E[Z|V, X = 0]P(X = 0|V)$$

Given $X = 1$ we have that $Z = Y_1$ and $E[Z|V, X = 1] = E[Y_1|V]$. Since $Y_1$ and $V$ are both gaussian, the conditional expectation is going to be a linear function of $V$:

$$E[Z|V, X = 1] = E[Y_1|V] = E[Y_1] + \rho \frac{\sigma_Z}{\sigma_V} (V - E[V])$$
and since $E[Y_1] = E[V] = -1$,

$$E[Z|V, X = 1] = E[Y_1] + \rho \frac{\sigma Y_1}{\sigma V} (V - E[V]) = -1 + \frac{\sigma^2}{\sigma^2 + \sigma_W^2} (V + 1)$$

Similarly, given $X = 0$ we have that $Z = Y_2$ and

$$E[Z|V, X = 0] = E[Y_2] + \rho \frac{\sigma Y_2}{\sigma V} (V - E[V]) = 1 + \frac{\sigma^2}{\sigma^2 + \sigma_W^2} (V - 1)$$

Next, we need to compute $q_V = P(X = 1|V) = 1 - P(X = 0|V)$. From Bayes’ rule we have that:

$$q_V = P(X = 1|V) = \frac{f_{V|X=1}P(X = 1)}{f_{V|X=1}P(X = 1) + f_{V|X=0}P(X = 0)} = \frac{\mathcal{N}(-1, \sigma^2 + \sigma_W^2)p}{\mathcal{N}(-1, \sigma^2 + \sigma_W^2)p + \mathcal{N}(1, \sigma^2 + \sigma_W^2)(1-p)}$$

So, for each value of $V$ we can compute the number $q_V$. Finally, we have

$$E[Z|V] = q_V \left[ -1 + \frac{\sigma^2}{\sigma^2 + \sigma_W^2} (V + 1) \right] + (1 - q_V) \left[ 1 + \frac{\sigma^2}{\sigma^2 + \sigma_W^2} (V - 1) \right]$$

**Problem 9.3**

Assume that $X$ and $Y$ are i.i.d. $N(0; 1)$. Calculate $E[(X + Y)^4|X - Y]$. 

**Solution:** $X + Y$ and $X - Y$ are linear function of independent Gaussian random variables, so they are jointly Gaussian random variables. We can verify that they are uncorrelated


It follows that $X + Y$ and $X - Y$ are independent because they are jointly Gaussian and uncorrelated. Hence,

$$E[(X + Y)^4|X - Y] = E((X + Y)^4) = E(X^4 + 4X^3Y + 6X^2Y^2 + 4XY^3 + Y^4) = 3 + 6 + 3 = 12$$

**Problem 9.4**

Let $X$ and $Y$ be independent $N(0; 1)$ random variables. Show that $W = X^2 + Y^2$ is an exponential with parameter $\frac{1}{2}$. That is, the sum of the squares of two i.i.d. zero-mean Gaussian random variables is exponentially distributed!
Solution: We calculate the characteristic function of $W$. We find

\[
E[e^{sW}] = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{s(x^2 + y^2)} f_{X,Y}(x, y) dx dy \\
= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{s(x^2 + y^2)} \frac{1}{2\pi} e^{-\frac{(x^2 + y^2)}{2}} dx dy \\
= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{(s - \frac{1}{2})(x^2 + y^2)} dx dy \\
= \frac{1}{2\pi} \int_{0}^{2\pi} \int_{0}^{+\infty} e^{(s - \frac{1}{2})r^2} r dr d\theta \\
= \int_{0}^{+\infty} e^{(s - \frac{1}{2})r^2} r dr = \int_{0}^{+\infty} \frac{1}{2(s - \frac{1}{2})} d \left[ e^{(s - \frac{1}{2})r^2} \right] dr \\
= \frac{1}{2(s - \frac{1}{2})} \left[ 0 - 1 \right] = \frac{1}{1 - 2s}
\]

which is the moment generating function of an exponential random variable with parameter $\frac{1}{2}$. 

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