UC Berkeley
Department of Electrical Engineering and Computer Science

EE 126: PROBABLITY AND RANDOM PROCESSES

Solutions 5
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Review problems

Problem 5.1
On any given day your golf score is any integer from 101 to 110, each with probability 0.1. Determined to improve your score, you decide to play with three balls and declare as your score the minimum Y of the scores X_1, X_2, and X_3 obtained with balls 1, 2, and 3, respectively.

Calculate the PMF of Y. By how much has your expected score improved as a result of using three balls?
Solution:

1. Think of the possible values of x_1, x_2 and x_3 as coordinates in a cubical lattice of side 10. Make a sketch of this cubical lattice, and count all of the points corresponding to $\min(x_1, x_2, x_3) = 101$. These points are trivial to count geometrically! They are the points in a cubical lattice of side 10, minus the points in a cubical lattice of side 9. Similarly, we can count the number of points corresponding to $\min(x_1, x_2, x_3) = (111 - k)$ as the number of points in a cubical lattice of side k, minus the number of points in a cubical lattice of side $(k - 1)$. Since this problem follows a discrete uniform law, this counting solution is sufficient to find the PMF:

$$p_x(k) = \begin{cases} \frac{(111-k)^3-(110-k)^3}{10^3}, & 100 < k \leq 110 \\ 0, & \text{otherwise} \end{cases}$$

Alternatively, the solution can be derived more generally using CDF's (it may help to read ahead a little to understand this solution method). We first determine the distribution of $x = \min(x_1, x_2, x_3)$:

$$\begin{aligned} F_x(k) &= P(x \leq k) \\ &= 1 - P(x > k) \\ &= 1 - P(x_1 > k, x_2 > k, x_3 > k) \\ &= 1 - P(x_1 > k)P(x_2 > k)P(x_3 > k) \quad \text{since } x_1, x_2, x_3 \text{ are independent} \\ &= \begin{cases} 0, & k \leq 100, \\ 1 - \left(\frac{110-k}{10}\right)^3, & 100 < k \leq 110 \\ 1, & k > 110 \end{cases} \end{aligned}$$

where we have used the fact that

$$P(x_i > k) = \begin{cases} 1, & k \leq 100, \\ \left(\frac{110-k}{10}\right), & 100 < k \leq 110 \\ 0, & k > 110 \end{cases}$$

Then we can obtain the PMF of x by the following method:

$$\begin{aligned}
p_x(k) &= F_x(k) - F_x(k-1) \\
&= \begin{cases} 0, & k \le 100, \\ 1 - \left(\frac{110-101}{10}\right)^3, & k = 101 \\ \left(\frac{111-k}{10}\right)^3 - \left(\frac{110-k}{10}\right)^3, & 101 < k \le 110 \\ 0, & k > 110 \end{cases} \\
&= \begin{cases} \frac{(111-k)^3 - (110-k)^3}{10^3}, & 100 < k \le 110 \\ 0, & \text{otherwise} \end{cases}
\end{aligned}$$

2. By symmetry, the expected value of the score on any particular ball is 105.5. This can also be seen using the formula for expected value:

$$E[x_i] = \sum_{k=101}^{110} k \cdot p_{x_i}(k) = \sum_{k=101}^{110} k \frac{1}{10} = 105.5$$

We now need to calculate the expected value of x and compare it to the above:

$$\begin{aligned}
E[x] &= \sum_{k=-\infty}^{\infty} k \cdot p_x(k) \\
&= \sum_{k=101}^{110} k \cdot p_x(k) \\
&= \sum_{k=101}^{110} k \frac{(111-k)^3 - (110-k)^3}{10^3} \\
&= 103.025
\end{aligned}$$

The expected improvement is therefore 105.5 - 103.025 = 2.475.

Problem 5.2

Suppose you and your friend play a game where each of you throws a 6-sided die, each of your throws being independent. Each time you play the game, if the largest of the two values you obtained from each die is greater than 4, then you win 1 dollar; otherwise, you lose 1 dollar. Suppose that you play the game $n \ge 3$ times, each game being independent of the others.

(a) What is the amount of money you expect to win on the first and last game combined?

(b) How much do you expect to win in your last game given that you lost in the first game?

(c) How much do you expect to have won in your last game given that you won the first game and you won a total of m dollars at the end?

(d) What is the probability that you won both the first and last game given that you won a total of m dollars at the end?

Solution:

(a) Let Y_i be the random variable taking values 1 and -1 depending on whether you won or lost the ith game. Let X_1 and X_2 be the random variables denoting the outcomes of the first die and second die respectively. On each game you win independently if the maximum between X_1 and X_2 is greater than 4. So,

$$\begin{aligned}
p = P(Y_i = 1) = P(\max(X_1, X_2) > 4) &= 1 - P(\max(X_1, X_2) \leq 4) \\
&= 1 - P(X_1 \leq 4, X_2 \leq 4) \\
&= 1 - P(X_1 \leq 4)^2 \\
&= 1 - \left(\frac{4}{6}\right)^2 \\
&= \frac{5}{9}
\end{aligned}$$

And,

$$E[Y_i] = p - (1-p) = 2p - 1 = \frac{1}{9}.$$

We need to find $E[Y_1 + Y_n]$. From the linearity of the expectation $E[Y_1 + Y_n] = E[Y_1] + E[Y_n] = 2\frac{1}{9} = \frac{2}{9}$

(b) We are asked to find $E[Y_n | Y_1 = -1]$. Since Y_n and Y_1 are independent random variables, $E[Y_n | Y_1 = -1] = E[Y_n] = \frac{1}{9}$.

(c) Let $Z = Y_1 + Y_2 + \ldots + Y_n$. We are asked to compute $E[Y_n | Y_1 = 1, Z = m]$. Notice that although Y_n and Y_1 are independent random variables, they are *not* independent given Z. We have that

$$E[Y_n | Y_1 = 1, Z = m] = P(Y_n = 1 | Y_1 = 1, Z = m) - P(Y_n = -1 | Y_1 = 1, Z = m)$$

and, using Bayes' rule for conditional probability we have

$$P(Y_n = 1 | Y_1 = 1, Z = m) = \frac{P(Z = m | Y_1 = 1, Y_n = 1) P(Y_n = 1)}{P(Z = m | Y_1 = 1)}$$

where

$$\begin{aligned}
P(Z = m | Y_1 = 1) &= P(Z = m | Y_1 = 1, Y_n = 1) P(Y_n = 1) \\
&\quad + P(Z = m | Y_1 = 1, Y_n = -1) P(Y_n = -1)
\end{aligned}$$

So, we just need to compute $P(Z = m | Y_1 = 1, Y_n = 1)$ and $P(Z = m | Y_1 = 1, Y_n = -1)$ (see next part for the explanation):

$$P(Z=m|Y_1=1,Y_n=1)=\binom{n-2}{\frac{1}{2}(n+m-4)}p^{\frac{1}{2}(n+m-4)}(1-p)^{\frac{1}{2}(n-m)}$$

$$P(Z=m|Y_1=1,Y_n=-1)=\binom{n-2}{\frac{1}{2}(n+m-2)}p^{\frac{1}{2}(n+m-2)}(1-p)^{\frac{1}{2}(n-m-2)}$$

if $\frac{1}{2}(n+m-4)$ and $\frac{1}{2}(n+m-2)$ are integers, otherwise the probabilities are 0.

From which,

$$\begin{aligned} P(Y_n=1|Y_1=1,Z=m) &= \frac{\binom{n-2}{\frac{1}{2}(n+m-4)}}{\binom{n-2}{\frac{1}{2}(n+m-4)}+\binom{n-2}{\frac{1}{2}(n+m-2)}} \\ P(Y_n=-1|Y_1=1,Z=m) &= \frac{\binom{n-2}{\frac{1}{2}(n+m-2)}}{\binom{n-2}{\frac{1}{2}(n+m-4)}+\binom{n-2}{\frac{1}{2}(n+m-2)}} \end{aligned}$$

And finally

$$E[Y_n|Y_1=1,Z=m] = \frac{\binom{n-2}{\frac{1}{2}(n+m-4)}-\binom{n-2}{\frac{1}{2}(n+m-2)}}{\binom{n-2}{\frac{1}{2}(n+m-4)}+\binom{n-2}{\frac{1}{2}(n+m-2)}}$$

(d) using Bayes' rule for conditional probability we have

$$P(Y_n=1,Y_1=1|Z=m) = \frac{P(Z=m|Y_1=1,Y_n=1)P(Y_n=1)P(Y_1=1)}{P(Z=m)}$$

Given that $Y_1=1, Y_n=1$, Z is equal to m if in the remaining $n-2$ games you win a total of $m-2$ dollars. This is possible only if you win $\frac{1}{2}(n-2+m-2)$ games and loose the remaining $\frac{1}{2}(n-2-m+2)$ games. In fact, the aggregate amount of dollars that you win is $\frac{1}{2}(n-2+m-2)-\frac{1}{2}(n-2-m+2)=m-2$. Thus, we have

$$P(Z=m|Y_1=1,Y_n=1)=\binom{n-2}{\frac{1}{2}(n+m-4)}p^{\frac{1}{2}(n+m-4)}(1-p)^{\frac{1}{2}(n-m)}$$

if $\frac{1}{2}(n+m-4)$ is an integer, otherwise the probability is 0.

We can compute $P(Z=m)$ using a similar argument: you play n times and need to win a total of m dollars starting from 0; so, you need to win $\frac{1}{2}(n+m)$ games and loose the remaining $\frac{1}{2}(n-m)$ games. In fact, the aggregate amount of dollars that you win is $\frac{1}{2}(n+m)-\frac{1}{2}(n-m)=m$. Thus,

$$P(Z=m)=\binom{n}{\frac{1}{2}(n+m)}p^{\frac{1}{2}(n+m)}(1-p)^{\frac{1}{2}(n-m)}$$

if $\frac{1}{2}(n+m)$ is an integer, otherwise the probability is 0.

Finally we obtain

$$
\begin{aligned}
P(Y_n = 1, Y_1 = 1 | Z = m) &= \frac{\binom{n-2}{\frac{1}{2}(n+m-4)} p^{\frac{1}{2}(n+m-4)} (1-p)^{\frac{1}{2}(n-m)} p^2}{\binom{n}{\frac{1}{2}(n+m)} p^{\frac{1}{2}(n+m)} (1-p)^{\frac{1}{2}(n-m)}} \\
&= \frac{\binom{n-2}{\frac{1}{2}(n+m-4)}}{\binom{n}{\frac{1}{2}(n+m)}}
\end{aligned}
$$

Problem 5.3

Consider the following game. You throw two fair coins independently. Let's refer to the coins as coin 1 and coin 2. If they have the same outcome, you win; otherwise, you lose.

(a) Is the outcome of coin 1 independent of whether you win or lose? Is the outcome of coin 2 independent of whether you win or lose? Explain your answer.

(b) Are the outcomes of the two coins independent of each other conditioned on your having won or lost? Explain your answer.

Solution:

(a) Yes. To see this, let X be a random variable that takes values 1 or 0 depending on whether the first coin came out head or tail, respectively. Similarly, let Y be a random variable that takes values 1 or 0 depending on whether the first coin came out head or tail, respectively. Finally, let W be a random variable that takes values 1 or 0 depending on whether you win or lose, respectively. We have

$$p_{W|X}(0|0) = p_Y(1) = 1/2 \text{ and } p_{W|X}(0|1) = p_Y(0) = 1/2.$$

Therefore,

$$p_W(0) = p_{W|X}(0|0)p_X(0) + p_{W|X}(0|1)pX(1) = (1/2)(1/2) + (1/2)(1/2) = 1/2.$$

Likewise,

$$p_{W|X}(1|0) = p_Y(0) = 1/2 \text{ and } p_{W|X}(1|1) = p_Y(1) = 1/2.$$

Therefore,

$$p_W(1) = p_{W|X}(1|0)p_X(0) + p_{W|X}(1|1)p_X(1) = (1/2)(1/2) + (1/2)(1/2) = 1/2.$$

Hence, we see that $p_W(w) = p_{W|X}(w|x)$ for all $w, x = 0, 1$, which establishes the independence of X and W. Showing that Y and W are independent is completely symmetric.

(b) No. To see this, note that once we know we have won, if in addition we were to know that the second coin came up heads, then we know that the first coin also came up heads. More formally, note that we have $p_{X|Y,W}(1|1, 1) = 1$, but, since X and W are independent, $p_{X|W}(1|1) = pX(1) = 1/2$.

Problem 5.4

Your computer has been acting very strangely lately, and you suspect that it might have a virus on it. Unfortunately, all 12 of the different virus detection programs you own are somewhat outdated. You know that if your computer really does have a virus, each of a programs, independently of the others, has a .8 chance of correctly identifying your computer to be infected, and a .2 chance of thinking your computer is fine. On the other hand, if your computer does not have a virus, each program has a .9 chance of believing you computer to be free from viruses, and a .1 chance of wrongly thinking your computer is infected. Given that your computer has a 65% chance of being infected with some virus, and given that you will believe your virus protection programs only if 9 or more of them agree, find the probability that your detection programs will lead you to the right answer.

Solution:

Let A denote the event that your detection programs lead you to the correct conclusion about your computer. Let V be the event that your computer has a virus, and let V^c be the event that your computer does not have a virus. Note that V, V^c are mutually exclusive and collectively exhaustive, and thus we can use the relation:

$$P(A) = P(A \cap V) + P(A \cap V^c)$$

where $P(A \cap V) = P(A|V) \cdot P(V)$ and similarly for V^c. Now, $P(A|V)$ and $P(A|V^c)$ can be found by the complement of the binomial distribution function. Thus we have:

$$\begin{aligned} P(A|V) &= \binom{12}{9} \cdot .8^9 \cdot .2^3 + \binom{12}{10} \cdot .8^{10} \cdot .2^2 + \binom{12}{11} \cdot .8^{11} \cdot .2^1 + \binom{12}{12} \cdot .8^{12} \cdot .2^0 \\ &= .7899 \end{aligned}$$

Similarly we find that $P(A|V^c) = .9742$ therefore:

$$P(A) = P(V) \cdot P(A|V) + P(V^c_\cdot P(A|V^c) = .8544.$$

Problem 5.5

Your filthy roommate, Jerome, is (finally) doing his laundry. He has so much to do that he is doing it over four separate days—Monday, Tuesday, Wednesday, and Thursday—and, on each day, he has five quarters in each of his left and right pockets for use in the coin-operated machines. When drawing a quarter from his pockets, he draws from his left pocket with probability p and from his right pocket with probability $1 - p$, independently of previous draws.

On Monday, Jerome draws five quarters from his pockets to get the \$1.25 required for a washing machine. It turns out that one quarter is drawn from his left pocket and four quarters are drawn from his right pocket. To begin with, one of the quarters in his left pocket is Canadian (and all the other nine are American).

(a) What is the probability that his Canadian quarter remains in his left pocket?

(b) What is the expectation and variance of the number of American quarters remaining in his left pocket?

(c) On Tuesday, Jerome again draws five quarters from his pockets. What is the probability that exactly three of the five quarters drawn are from his left pocket?

(d) On Wednesday, Jerome decides to use the extra-large capacity washing machine, which costs \$2.00; so he draws eight quarters from his pockets. It turns out that the eighth quarter drawn is the fifth drawn from his left pocket, leaving it empty. What is probability of this event?

(e) On Thursday, Jerome does not draw a fixed number of quarters from his pockets, but instead draws quarters until one of his pockets is empty. Let X be the number of quarters remaining in his other pocket. Let A be the event that Jerome's left pocket empties before his right pocket. Find $\mathbb{P}(\{X = x\} \cap A)$ for $x = 1, 2, \ldots, 5$.

Solution:

(a) One quarter is drawn from Jeromes left pocket. The probability that it is the Canadian one is 1/5, so Jeromes Canadian quarter remains in his left pocket with probability 4/5.

(b) Let U be the number of Canadian quarters remaining in Jeromes left pocket, and let V be the number of American ones. Hence $p_U(0) = 1/5$ and $p_U(1) = 4/5$. Therefore, $E[U] = E[U2] = 4/5$, so $\text{var}(U) = 4/5 - (4/5)2 = 4/25$. Now, $V = 4 - U$ and it follows that $E[V] = 4 + E[-U] = 4 - 4/5 = 16/5$, and that $\text{var}(V) = \text{var}(4) + \text{var}(-U) = \text{var}(U) = 4/25$.

(c) The question asks for the probability of three successes in five independent Bernoulli trials, where a success here is a draw from Jeromes left pocket. Therefore, the probability in question is $\binom{5}{3}p^3(1-p)^2$.

(d) Let A denote the event that the the last quarter drawn is the fifth drawn from the left pocket, and B the event that Jerome draws eight quarters from his pockets. We are asked to compute
$$P(A|B) = \frac{P(A \cap B)}{P(B)}.$$
In order to collect a total of 8 coins, Jerome needs to select at least 3 coins from the left pocket (and 5 coins from the right pocket), but he cannot select more than 5 coins from the left pocket. So, the total probability of all the possible sequences of eight drawing from the two pockets is
$$P(B) = \binom{8}{3}p^3(1-p)^5 + \binom{8}{4}p^4(1-p)^4 + \binom{8}{5}p^5(1-p)^3$$
Among all these possible ways of collecting the 8 coins, we are interested in the specific sequence that has 5 coins drawn from from the left pocket (and 3 coins from the right pocket), and the last position in the sequence is occupied by a coin from the left pocket. So we have
$$P(A \cap B) = \binom{1}{1}\binom{8-1}{4}p^5(1-p)^3$$

Finally, we have

$$P(A|B) = \frac{p^2}{\frac{8}{20}(1-p)^2 + \frac{8}{16}p(1-p) + \frac{8}{20}p^2}$$

(e) Notice that for this part of the problem, the underlying allowable sequences are different from the previous part. In fact, in part (d) a *fixed* number 8 of trials take place, whereas in this part trials are taken until one pocket is empty. This implies that for this part of the problem Jerome's drawing can be modeled as a Bernoulli trial, where with probability p the quarter is drawn from the left pocket and with probability $1-p$ from the left pocket. At this point it is clear that, for $x = 1, 2, ..., 5$, the event $X = x \cap A$ is equivalent to the event that the $(10-x)th$ quarter drawn is the fifth drawn from Jeromes left pocket. Hence,

$$P(X = x \cap A) = \binom{9-x}{4} p^4 (1-p)^{5-x} p$$