UC Berkeley
Department of Electrical Engineering and Computer Science

EE 126: PROBABLITY AND RANDOM PROCESSES

Problem Set 5
Spring 2006

Issued: Friday, February 24, 2006 **Due:** Friday, March 3, 2006

Reading: Bertsekas & Tsitsiklis, §3.1—§3.4.

Problem 5.1
(Quantization) An analog-to-digital device converts a continuous signal into a discrete set of values. For example, the intensity in images are frequently converted to 8 bits, or $256 = 2^8$ levels of resolution, in a process known as quantization. As a particular example of a quantizer, consider the "greatest integer" function $\lfloor y \rfloor$, which maps the real number y to the largest integer $\leq y$, so that for example $\lfloor 3.85 \rfloor = 3$. Supposing that X is exponentially distributed with rate $\lambda = 1$, compute $\mathbb{E}[\,\lfloor X \rfloor]$.

Problem 5.2
A random variable X is called a *shifted exponential* when its PDF has the following form

$$f_X(x) = \frac{1}{\theta} e^{-(x-\alpha)/\theta}, \quad \alpha < x < \infty$$

(a) Find the CDF of X.

(b) Calculate the mean and variance of X.

(c) Find the real number m that satisfies $F_X(m) = \frac{1}{2}$. This number m is called the *median* of the random variable.

Problem 5.3
Consider the continuous random variable X with PDF:

$$f_X(x) = \left\{ \begin{array}{ll} \frac{1}{8} & 1 \leq x < 2 \text{ or } 3 \leq x < 4 \\ \frac{3}{4} & 2 \leq x < 3 \\ 0 & \text{otherwise.} \end{array} \right\}$$

Let X_i denote the experimental value of random variable X on trial i. All X_i's are independent of each other. Find the expected number of trials before a value greater than $\frac{7}{2}$ is observed. Your answer should be a number.

Problem 5.4
A signal $s = 3$ is transmitted from a satellite but is corrupted by noise, and the received signal is $X = s + W$. When the weather is good, which happens with probability 2/3, W is normal with zero mean and variance 4. When the weather is bad, W is normal with zero mean and variance 9. In the absence of any weather information:

(a) Compute the PDF of X.

(b) Calculate the probability that X is between 2 and 4.

Problem 5.5

(a) Let X be a zero-mean Gaussian random variable with variance σ^2. Derive an expression for $\mathbf{E}[X^n]$, the nth moment, that is valid for all $n \geq 0$. *Hint*: Relate the nth moment to the $(n-2)$th moment. Separately consider the cases of n even and n odd.

(b) A *central Chi-Squared* random variable Y, with n degrees of freedom, is defined as

$$Y = X_1^2 + X_2^2 + \cdots + X_n^2,$$

where the X_i's are *independent and identically distributed* zero-mean Gaussian random variables with variance σ^2. Find the expectation and variance of Y.

Problem 5.6

One of two wheels of fortune, A and B, is selected by the flip of a fair coin, and the wheel chosen is spun once to determine the experimental value of random variable X. When wheel A is spun, its value is the random variable Y; when B is spun, its value is the random variable W. These two RVS are described by the PDFs

$$f_Y(y) = \begin{cases} 1, & 0 < y \leq 1; \\ 0, & \text{otherwise,} \end{cases} \qquad \text{and} \qquad f_W(w) = \begin{cases} 3, & 0 < w \leq \frac{1}{3}; \\ 0, & \text{otherwise.} \end{cases}$$

If we are told the experimental value of X was less than $\frac{1}{4}$, what is the conditional probability that wheel A was the one selected?