UC Berkeley
Department of Electrical Engineering and Computer Science

EE 126: PROBABLITY AND RANDOM PROCESSES

**Problem Set 6**
Spring 2006

**Issued:** Friday, March 3, 2006 **Due:** Friday, March 10, 2006

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**Reading:** Bertsekas & Tsitsiklis, §3.4—§3.6.

**Problem 6.1**
Beginning at time $t = 0$ we begin using bulbs, one at a time, to illuminate a room. Bulbs are replaced immediately upon failure. Each new bulb is selected independently by an equally likely choice between a bulb from company A and a bulb from company B. The lifetime of any particular bulb is exponential with mean 1 (for type A bulbs), and mean 3 (for type B bulbs).

(a) Find $\mathbb{P}(D)$, the probability that there are no bulb failures during the first $T$ hours of this process.

(b) Given that there are no failures during the first $T$ hours of this process, determine $\mathbb{P}(C_{1A}|D)$, the conditional probability that the first bulb used is a bulb from company A.

(c) Determine the expected value and variance for $X$, the time until the first bulb failure.

**Problem 6.2**
Suppose $n$ runners run a race. At one point, all the runners are uniformly distributed on a stretch of one mile that starts at point $A$ and ends at point $B$. Number the runners from 1 to $n$, according to the order in which they are at this particular moment. Let $X_i$ denote the distance between point $A$ and the $i$th runner and let $X_0 = 0$ and $X_{n+1} = 1$.

(a) For $n = 2$, find $\mathbb{P}(X_i > X_{i-1} + t)$ for $i = 1, 2, 3$.

(b) For $n \geq 1$, find $\mathbb{P}(X_i > X_{i-1} + t)$ for $i = 1, 2, \ldots, n + 1$.
(Hint: The answer does not depend on $i$.)

**Problem 6.3**
*(Optional; not to be graded)* Consider the following problem and a purported solution. Either declare the solution to be correct or explain the flaw.

*Question:* Let $X$ and $Y$ have the joint density

$$f_{X,Y}(x, y) = \begin{cases} 1, & x \in [0, 1] \quad \text{and} \quad y \in [x, x + 1]; \\ 0, & \text{otherwise.} \end{cases}$$

Find $f_X(x)$, $f_Y(y)$, and $f_{Y|X}(y|x)$. Are $X$ and $Y$ independent?

*Solution:*

$$f_X(x) = \int f_{X,Y}(x, y)\, dy = \int_x^{x+1} 1 \cdot dy = 1.$$

$$f_Y(y) = \int f_{X,Y}(x,y)\,dx = \int_0^1 1 \cdot dx = 1.$$

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{1}{1} = 1.$$

Since $f_{Y|X}(y|x)$ does not depend on $y$, we have that $X$ and $Y$ are independent. Alternatively, $X$ and $Y$ are independent because $f_{X,Y}(x,y) = f_X(x)f_Y(y)$.

**Problem 6.4**
A group of $N$ archers shoot at a target. The distance of each shot from the center of the target is uniformly distributed between 0 to 1, independently of the other shots.

(a) Find the expected distance from the winner's arrow to the center.

(b) Find the expected distance from the loser's arrow to the center (his arrow is the arrow farthest away from the origin).

**Problem 6.5**
A river has a bridge every 8 miles. You would like to cross the river by bridge, but you are currently on a road that runs parallel to the river and is separated from the river by a dense forest. Occasionally there is a straight path through the forest which runs perpendicular to the river and the road. Assume you start walking on the road at a point such that after walking 2 miles, you are directly opposite a bridge (so if you found a path after walking 2 miles, you would emerge from the forest at a bridge). Let $R$ be the distance you walk on the road before you find a path through the forest. Let $S$ be the distance you walk on the river bank to reach a bridge, assuming you head for the nearest bridge when you emerge from the forest. Find the PDF of $S$ for the following cases.

(a) the PDF $f_R(r)$ is nonzero in the interval $[0, 12]$ and zero everywhere else.

(b) the random variable $R$ is exponentially distributed with mean 1 mile.

**Problem 6.6**
Find the PDF of $X^2 + 2X + 1$ in terms of the PDF of $X$. Then specialize the answer to the case where $X$ is uniformly distributed between 0 and 1.

**Problem 6.7**
Let $X$ be a non-negative random variable with PDF $f_X$ and CDF $F_X$ with a finite expectation $\mathbb{E}[X]$.

(a) For any $x \geq 0$, show that $x\,[1 - F_X(x)] \leq \int_x^\infty t\, f_X(t)\,dt$. Hence conclude that $x\,[1 - F_X(x)] \to 0$ as $x \to +\infty$.

(b) Use part (a) to show that $\mathbb{E}[X] = \int_0^\infty [1 - F_X(x)]\,dx$. (*Hint:* Consider integration by parts.)

(c) Use part (b) to show that if $X$ is an exponential RV with parameter $\lambda$, then $\mathbb{E}[X] = 1/\lambda$.