UC Berkeley
Department of Electrical Engineering and Computer Science

EE 126: PROBABLITY AND RANDOM PROCESSES

Problem Set 7
Spring 2006

Issued: Friday, March 10, 2006 **Due:** Friday, March 17, 2006

Reading: Bertsekas & Tsitsiklis, §3.6, §4.1, 4.2

Problem 7.1

(a) The random variable X is exponentially distributed with parameter $\lambda = 1$. Derive the PDF $f_Z(z)$, where $Z = e^{3X}$.

(b) Random variables X and Y have the joint PDF

$$f_{X,Y}(x,y) = \begin{cases} c & , \text{ if } 0 \leq y < x \leq 2 \\ 0 & , \text{ otherwise} \end{cases} .$$

Derive the PDF $f_Z(z)$, where $Z = \frac{Y}{X}$.

Problem 7.2

The transform and the mean of a random variable X are given by $M_X(s) = ae^s + be^{13(e^s - 1)}$ and $E[X] = 5$ respectively. Determine the numerical values of:

(a) The constants a and b.

(b) $\mathbb{E}[e^{5X}]$.

(c) $\mathbb{P}(X = 1)$.

(d) $\mathbb{E}[X^2]$.

Problem 7.3

Xavier and Yolanda enter a frisbee-throwing contest. The distance (in meters) that Xavier throws is uniformly distributed between 0 and 100, and the distance that Yolanda throws (in meters) is exponential with $\lambda = 1/60$.

(a) What is the probability that Xavier's frisbee lands at 75m?

(b) What is the probability that Yolanda throws further than 100m?

(c) What is the expected distance of each competitor's throw?

(d) Which of the two competitors is more likely to throw further?

(e) Given that Xavier's frisbee lands at 75m, find PDF for the distance of Yolanda's throw.

(f) Let W be the distance Yolanda's frisbee lands past Xavier's. Find the PDF of W.

Problem 7.4

Let continuous random variables X, Y and Z be independent and identically distributed according to the uniform distribution in the unit interval $[0, 1]$. Consider two new random variables defined by $V = XY$ and $W = Z^2$. Derive the joint PDF $f_{V,W}(v, w)$, and show that $\mathbb{P}(XY < Z^2) = \frac{5}{9}$.

Problem 7.5

(Optional; not to be graded) Independent random variables X and Y have PDFs whose transforms are $M_X(s)$ and $M_Y(s)$. Random variable R is defined to be $R = X + Y$. Use $M_R(s)$ and the moment generating properties of transforms to show that $E[R] = E[X]+E[Y]$ and $\mathrm{var}(R) = \mathrm{var}(X) + \mathrm{var}(Y)$.

Problem 7.6

A coin is tossed repeatedly, heads appearing with probability q on each toss. Let random variable T denote the total number of tosses after which a run of n consecutive heads has appeared for the *first* time.

(a) Show that the PMF for T can be expressed as

$$p_T(k) = \begin{cases} 0 & , \quad k < n \\ q^n & , \quad k = n \\ \left(\sum_{i=k-n}^{\infty} p_T(i) \right) (1-q)q^n & , \quad k \geq n+1 \end{cases} .$$

(b) Determine the moment generating function $M_T(s)$ and compute $\mathbb{E}[T]$. (*Hint:* Your answer for $n = 1$ should be linked to a geometric random variable.)

Problem 7.7

Steve is trying to decide how to invest his wealth in the stock market. He decides to use a probabilistic model for changes in share prices. He believes that, at the end of the day, the change of price Z_i of a share of a particular company i is the sum of two components: X_i, due solely to the performance of the company, and the other Y due to investors' jitter. Suppose that $Y \sim N(0, 1)$ and is independent of X_i.

(a) If X_1 is Gaussian with a mean of 1 dollar and variance equal to 4, compute the PDF of Z_1.

(b) If X_2 is equal to -1 dollars with probability 0.5, and 3 dollars with probability 0.5, compute the PDF of Z_2.

(c) If X_3 is uniformly distributed between -2.5 dollars and 4.5 dollars, compute the PDF of Z_3. (No closed form expression is necessary.)

(d) Being risk averse, Steve now decides to invest only in the first two companies. He uniformly chooses a portion V of his wealth to invest in company 1 (V is uniform between 0 and 1.) Assuming that a share of company 1 or 2 costs 100 dollars, what is the expected value of the relative increase/decrease of his wealth?