

Discussion 10 Solutions
March 22, 2012

1. (a) K has a Poisson distribution with average arrival time $\mu = \lambda_c T$

$$p_K(k) = \frac{(\lambda_c T)^k e^{-\lambda_c T}}{k!}, \quad k = 0, 1, 2, \dots; T \geq 0.$$

(b) i. $\mathbf{P}(\text{conscious response}) = \left(\frac{\lambda_c}{\lambda_c + \lambda_s} \right).$

ii. $\mathbf{P}(\text{conscious correct response}) = \mathbf{P}(\text{conscious resp}) \mathbf{P}(\text{correct resp} | \text{conscious resp}) = \left(\frac{\lambda_c}{\lambda_c + \lambda_s} p_c \right).$

- (c) Since the conscious and subconscious responses are generated independently,

$$\begin{aligned} & \mathbf{P}(r \text{ conscious responses and } s \text{ subconscious responses in interval } T) \\ &= \mathbf{P}(r \text{ conscious responses in } T) \mathbf{P}(s \text{ unconscious responses in } T) \\ &= \frac{(\lambda_c T)^r e^{-\lambda_c T}}{r!} \cdot \frac{(\lambda_s T)^s e^{-\lambda_s T}}{s!} \end{aligned}$$

- (d) Let X_s = the time from the start of the exam to the time of the 1st subconscious response, and X_c = the time from the 1st subconscious response to the time of the next conscious response.

Note that X_s and X_c are independent exponentially distributed random variables with parameters λ_s and λ_c , respectively.

$$\begin{aligned} f_{X_s}(x_s) &= \lambda_s e^{-\lambda_s x_s} \text{ when } x_s \geq 0 \\ &= 0 \text{ otherwise} \\ f_{X_c}(x_c) &= \lambda_c e^{-\lambda_c x_c} \text{ when } x_c \geq 0 \\ &= 0 \text{ otherwise} \end{aligned}$$

$X = X_s + X_c$. So its PDF is the convolution of the two exponential distributions. For $x \geq 0$

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{\infty} \lambda_s e^{-\lambda_s(x-x_c)} \lambda_c e^{-\lambda_c x_c} dx_c \\ &= \int_0^x \lambda_s \lambda_c e^{-\lambda_s x} e^{(\lambda_s - \lambda_c)x_c} dx_c \text{ because } x - x_c > 0 \\ &= \lambda_s \lambda_c e^{-\lambda_s x} \int_0^x e^{(\lambda_s - \lambda_c)x_c} dx_c \\ &= \frac{\lambda_s \lambda_c}{\lambda_s - \lambda_c} e^{-\lambda_s x} (e^{(\lambda_s - \lambda_c)x} - 1) \text{ because } \lambda_s \neq \lambda_c \\ &= \frac{\lambda_s \lambda_c}{\lambda_s - \lambda_c} (e^{-\lambda_c x} - e^{-\lambda_s x}) \end{aligned}$$

2. (a) Since we are looking for the number of “trials” up to and including the first “success,” N is a geometric random variable with parameter p .

$$p_N(n) = (1 - p)^{n-1} p, \quad n \geq 1.$$

(b) The length of time spent driving to each intersection is exponentially distributed with parameter λ . Since the probability of Shem observing an accident at a given intersection is p , the distribution of the length of time in between accident reports is exponential but with parameter $p\lambda$ (think of Poisson splitting). Thus,
 $f_Q(q) = (p\lambda)e^{-qp\lambda}, q \geq 0.$

(c) Since the interarrival time of accidents is exponentially distributed with parameter $p\lambda$, the number of arrivals in a given amount of time τ is a Poisson random variable with parameter $p\lambda\tau$. Thus,

$$\mathbf{P}(m \text{ arrivals in 2 hours}) = p_M(m) = \frac{e^{-2p\lambda}(2p\lambda)^m}{m!}, \quad m \geq 0.$$

(d) We can view the radio calls to Shem and the accident reports as independent Poisson processes with arrival rates μ and $p\lambda$, respectively. When the two independent Poisson processes are joined, the resultant is a Poisson process with arrival rate $\mu + p\lambda$. Furthermore, the probability of an arrival from the radio calls is $\frac{\mu}{\mu + p\lambda}$. Since we are interested in the number of reported accidents between two radio calls, we can view this as a shifted Geometric random variable with parameter $\frac{\mu}{\mu + p\lambda}$. Thus,

$$p_K(k) = \left(\frac{p\lambda}{\mu + p\lambda}\right)^k \left(\frac{\mu}{\mu + p\lambda}\right), \quad k \geq 0.$$

(e) If we begin to observe Shem's radio calls at some random instant in time, due to the memoryless property of Poisson interarrivals, the distribution until he receives the next call will still be exponential with parameter μ . Also, the time from the previous call until the point at which we begin to observe Shem is also an exponential distribution with parameter μ . Thus, $W = X_1 + X_2$, where X_1 and X_2 have exponential distributions, i.e. W is a second order Erlang PDF.

$$f_W(w) = (\mu)^2 w e^{-w\mu}$$

3. See problem 6.27, page 337 in the textbook.