1. (a) Let \( X_i \) be a random variable indicating the quality of the \( i \)th bulb ("1" for good bulbs, "0" for bad ones). \( X_i \)'s are independent Bernoulli random variables. Let \( Z_n = \frac{X_1 + X_2 + \ldots + X_n}{n} \).

\[
E[Z_n] = p \quad \text{var}(Z_n) = \frac{n \text{var}(X_i)}{n^2} = \frac{\sigma^2}{n},
\]

where \( \sigma^2 \) is the variance of \( X_i \).

Applying Chebyshev’s inequality yields,

\[
P(|Z_n - p| \geq \epsilon) \leq \frac{\sigma^2}{n \epsilon^2},
\]

As \( n \to \infty \), \( \frac{\sigma^2}{n} \to 0 \) and \( P(|Z_n - p| \geq \epsilon) \to 0. \)

Hence, \( Z_n \) converges to \( p \) in probability.

(b) By Chebychev’s inequality,

\[
P(|Z_{50} - p| \geq 0.1) \leq \frac{\sigma^2}{50(0.1)^2},
\]

Since \( X_i \) is a Bernoulli random variable, its variance \( \sigma^2 \) is \( p(1 - p) \), which is less than or equal to \( \frac{1}{4} \). Thus,

\[
P(|Z_{50} - p| \geq 0.1) \leq \frac{1/4}{50(0.1)^2} = 0.5
\]

(c) By Chebychev’s inequality,

\[
P(|Z_n - p| \geq 0.1) \leq \frac{\sigma^2}{n \epsilon^2} \leq \frac{1/4}{n(0.1)^2}
\]

To guarantee a probability 0.95 of falling in the desired range,

\[
\frac{1/4}{n(0.1)^2} < 0.05,
\]

which yields \( n \geq 500. \) Note that \( n \geq 500 \) guarantees the accuracy specification even for the highest variance, namely 1/4. For smaller variances, we need smaller values of \( n \) to guarantee the desired accuracy. For example, if \( \sigma^2 = 1/16 \), \( n \geq 125 \) would suffice.

2. (a) \( E[X_n] = 0 \cdot (1 - \frac{1}{n}) + 1 \cdot \frac{1}{n} = \frac{1}{n} \)

\[
\text{var}(X_n) = (0 - \frac{1}{n})^2 \cdot (1 - \frac{1}{n}) + (1 - \frac{1}{n})^2 \cdot (\frac{1}{n}) = \frac{n-1}{n^2}
\]

\( E[Y_n] = 0 \cdot (1 - \frac{1}{n}) + n \cdot \frac{1}{n} = 1 \)

\[
\text{var}(Y_n) = (0 - 1)^2 \cdot (1 - \frac{1}{n}) + (n - 1)^2 \cdot (\frac{1}{n}) = n - 1
\]
(b) Using Chebyshev’s inequality, we have
\[ \lim_{n \to \infty} P(|X_n - \frac{1}{n}| \geq \epsilon) \leq \lim_{n \to \infty} \frac{n - 1}{n^2 \epsilon^2} = 0 \]
Moreover, \( \lim_{n \to \infty} \frac{1}{n} = 0 \).
It follows that \( X_n \) converges to 0 in probability. For \( Y_n \), Chebyshev suggests that,
\[ \lim_{n \to \infty} P(|Y_n - 1| \geq \epsilon) \leq \lim_{n \to \infty} \frac{n - 1}{\epsilon^2} = \infty, \]
Thus, we cannot conclude anything about the convergence of \( Y_n \) through Chebychev’s inequality.

(c) For every \( \epsilon > 0 \),
\[ \lim_{n \to \infty} P(|Y_n| \geq \epsilon) \leq \lim_{n \to \infty} \frac{1}{n} = 0, \]
Thus, \( Y_n \) converges to zero in probability.

(d) The statement is false. A counter example is \( Y_n \). It converges in probability to 0 yet its expected value is 1 for all \( n \).

(e) Using the Markov bound, we have
\[ P(|X_n - c| \geq \epsilon) = P(|X_n - c|^2 \geq \epsilon^2) \leq \frac{E[(X_n - c)^2]}{\epsilon^2} \]
Taking the limit as \( n \to \infty \), we obtain
\[ \lim_{n \to \infty} P(|X_n - c| \geq \epsilon) = 0, \]
which establishes convergence in probability.

(f) A counter example is \( Y_n \). \( Y_n \) converges to 0 in probability, but
\[ E[(Y_n - 0)^2] = 0 \cdot \left(1 - \frac{1}{n}\right) + (n^2) \cdot \frac{1}{n} = n \]
Thus,
\[ \lim_{n \to \infty} E[(Y_n - 0)^2] = \infty, \]
and \( Y_n \) does not converge to 0 in the mean square.

3. (a) No. Since \( X_i \) for any \( i \geq 1 \) is uniformly distributed between -1.0 and 1.0.
(b) Yes, to 0. Since for \( \epsilon > 0 \),
\[ \lim_{i \to \infty} P(|Y_i - 0| > \epsilon) = \lim_{i \to \infty} P\left(\frac{|X_i|}{i} > \epsilon\right) = \lim_{i \to \infty} [P(X_i > i\epsilon) + P(X_i < -i\epsilon)] = 0. \]
(c) Yes, to 0. Since for \( \epsilon > 0 \),
\[ \lim_{i \to \infty} P(|Z_i - 0| > \epsilon) = \lim_{i \to \infty} P\left(|(X_i)^i| > \epsilon\right) = \lim_{i \to \infty} \left[P(X_i > \epsilon^\frac{1}{i}) + P(X_i < -(\epsilon)^\frac{1}{i})\right] = \lim_{i \to \infty} \left[\frac{1}{2}(1 - \epsilon^\frac{1}{i}) + \frac{1}{2}(1 - \epsilon^\frac{1}{i})\right] = \lim_{i \to \infty} (1 - \sqrt{\epsilon}) = 0. \]