1. There are $n$ fish in a lake, some of which are green and the rest blue. Each day, Helen catches 1 fish. She is equally likely to catch any one of the $n$ fish in the lake. She throws back all the fish, but paints each green fish blue before throwing it back in. Let $G_i$ denote the event that there are $i$ green fish left in the lake.

(a) Show how to model this fishing exercise as a Markov chain, where $\{G_i\}$ are the states. Explain why your model satisfies the Markov property.

(b) Find the transition probabilities $\{p_{ij}\}$.

(c) List the transient and the recurrent states.

2. Consider the following Markov chain, with states labelled $s_0, s_1, \ldots, s_5$:

Given that the above process is in state $s_0$ just before the first trial, determine:

(a) The probability that the process enters $s_2$ for the first time as the result of the $k^{th}$ trial.

(b) The probability that the process never enters $s_4$.

(c) The probability that the process enters $s_2$ and then leaves $s_2$ on the next trial.

(d) The probability that the process enters $s_1$ for the first time on the third trial.

(e) The probability that the process is in state $s_3$ immediately after the $n^{th}$ trial.

(f) The long-term proportion of time spent in each state, $\pi_i$ for $i = 0, \ldots, 5$.

(g) The expected time until the process is absorbed.

**Solutions:**

1. (a) The number of remaining green fish at time $t$ completely determines all the relevant information of the system's entire history (relevant to predicting the future state.) Therefore it is immediate that the number of green fish is the state of the system and the process has the Markov property.
(b) At any given time, the number of green fish must either remain the same or decrease by 1. It decreases by 1 if a green fish is caught, which occurs with probability \( \frac{i}{n} \), and otherwise remains the same. Thus we have:

\[
p_{ij} = \begin{cases} 
\frac{n-i}{n} & j = i \\
\frac{i}{n} & j = i - 1 \\
0 & \text{otherwise}
\end{cases}
\]

(c) The state 0 is an absorbing state since there is a positive probability that the system will enter it, and once it does, it will remain there forever. Therefore the state with 0 green fish is the only recurrent state, and all other states are then transient.

2. (a) Let \( A_k \) be the event that the process enters \( s_2 \) for the first time on trial \( k \). The only way to enter state \( s_2 \) fot eh first time on the \( k \)th trial is to enter state \( s_3 \) on the first trial, remain in \( s_3 \) for the next \( k - 2 \) trials, and finally enter \( s_2 \) on the \( k \)th trial. Thus

\[
P(A_k) = p_{03} \cdot p_{33}^{k-2} \cdot p_{32} = \left( \frac{1}{3} \right) \left( \frac{1}{4} \right)^{k-2} \left( \frac{1}{4} \right) = \left( \frac{1}{3} \right) \left( \frac{1}{4} \right)^{k-1} \quad \text{for } k = 2, 3, \ldots
\]

(b) There are 3 possible ways for the process to never enter \( s_4 \). The first two are immediate transitions into the absorbing states, \( s_1 \) and \( s_5 \). The other is if the first transition is to \( s_3 \), and then there is an eventual transition to \( s_2 \). We will call this event \( A \). The probabilities of the first two cases are each \( \frac{1}{3} \). We can compute the probability of the event \( A \) in two different ways.

The first is to notice that \( A = \bigcup_{k=2}^{\infty} A_k \). As a result, we can compute

\[
P(A) = \sum_{k=2}^{\infty} P(A_k) = \sum_{k=2}^{\infty} \left( \frac{1}{3} \right) \left( \frac{1}{4} \right)^{k-1} = \frac{1}{12} \sum_{i=0}^{\infty} \left( \frac{1}{4} \right)^i = \left( \frac{1}{12} \right) \left( \frac{1}{1 - \frac{1}{4}} \right) = \frac{1}{9}.
\]

The second is to notice that given that the chain is in \( s_3 \), it must eventually either go to \( s_2 \) or \( s_4 \), so the probability that it eventually goes to \( s_2 \) must be

\[
P(A|\text{in } s_3) = \frac{p_{32}}{p_{32} + p_{34}} = \frac{1/4}{1/4 + 1/2} = \frac{1}{3}.
\]

Then we have

\[
P(A) = P(\text{go to } s_3)P(A|\text{in } s_3) = p_{03} \frac{1}{3} = \frac{1}{9}.
\]

So the final answer is

\[
P(\text{never go to } s_4) = p_{01} + p_{05} + P(A) = \frac{7}{9}.
\]

(c)

\[
P(\text{enter } s_2 \text{ and leave immediately}) = P(\text{enter } s_2)P(\text{leave immediately}|\text{in } s_2) = P(A) \cdot p_{21} = \left( \frac{1}{9} \right) \left( \frac{1}{2} \right) = \frac{1}{18}.
\]

(d) This event can only happen if the sequence of state transitions is:

\[
s_0 \rightarrow s_3 \rightarrow s_2 \rightarrow s_1
\]

Thus,

\[
P(\text{enter } s_1 \text{ for the first time on third trial}) = p_{03} \cdot p_{32} \cdot p_{21} = \left( \frac{1}{3} \right) \left( \frac{1}{4} \right) \left( \frac{1}{2} \right) = \frac{1}{24}.
\]
(e)  
\[ P(\text{in } s_3 \text{ at } n\text{th trial}) = P(\text{enter } s_3 \text{ on first trial})P(\text{stay in } s_3 \text{ for } n-1 \text{ trials}|\text{in } s_3) \]
\[ = p_{03}(p_{33})^{n-1} \]
\[ = \frac{1}{3} \left( \frac{1}{4} \right)^{n-1} \text{ for } n = 1, 2, \ldots \]

(f) States \( s_0, s_2, s_3, s_4 \) are transient, so
\[ \pi_0 = \pi_2 = \pi_3 = \pi_4 = 0. \]

We can also observe that
\[ \pi_1 = P(\text{eventually go to } s_1) = p_{01} + P(A) = \frac{1}{3} + \frac{1}{9} = \frac{4}{9}. \]

All the remaining probability must then go to \( \pi_5 \) giving
\[ \pi_5 = 1 - \pi_1 = \frac{5}{9}. \]

(g) Using the notation of the textbook:
\[ \mu_1 = \mu_5 = 0 \]
\[ \mu_2 = 1 + \frac{1}{2}\mu_1 + \frac{1}{2}\mu_2 = 2 \]
\[ \mu_4 = 1 + \frac{1}{2}\mu_4 + \frac{1}{2}\mu_5 = 2 \]
\[ \mu_3 = 1 + \frac{1}{4}\mu_2 + \frac{1}{4}\mu_3 + \frac{1}{2}\mu_4 = \frac{10}{3} \]
\[ \mu_0 = 1 + \frac{1}{3}\mu_1 + \frac{1}{3}\mu_3 + \frac{1}{3}\mu_5 = \frac{25}{9} \]

So the expected time to absorption is \( \frac{25}{9} \) time steps.