University of California - Berkeley
Department of Electrical Engineering & Computer Sciences
EE126 Probability and Random Processes
(Spring 2012)

Discussion 8 Notes
March 8, 2012

1. Two sticks of length l is cut into 3 peices, each by a different methods:

   - For the first stick, two locations on the stick are chosen idenpendently and uniformly at random. The stick is cut at those locations.
   - For the second stick, one location is chosen uniformly at random for the first cut. A fair coin is then flipped to choose one of the resulting pieces to cut again. The chosen piece is then cut at a location chosen uniformly at random.

   Let X denote the length of the leftmost piece of first stick, and let Y deonte the leftmost piece of the second stick.
   Compute $E[X]$, $var(x)$, $E[Y]$, and $var(Y)$.

Solutions:

1. For the first stick, denote the locations of the cuts A and B. Both are uniform over $[0, l]$. The length X is then $min(A, B)$. We can compute the CDF of X as:

$$\begin{aligned} F_X(x) = P(X \leq x) &= P(A \leq x \cup B \leq x) \\ &= 1 - P(A \leq x \cap B \leq x) \\ &= 1 - P(A > x)P(B > x) \\ &= 1 - \left(1 - \frac{x}{l}\right)^2 \\ &= 2\frac{x}{l} - \frac{x^2}{l^2} \end{aligned}$$

   Taking the derivative we get the PDF

$$\begin{aligned} f_X(x) = \frac{d}{dx}F_X(x) &= \frac{d}{dx}\left(2\frac{x}{l} - \frac{x^2}{l^2}\right) \\ &= \frac{2}{l} - \frac{2x}{l^2} \end{aligned}$$

   From the PDF we can compute the mean and variance.

$$E[X] = \int_0^l x f_X(x) dx$$

$$
\begin{aligned}
&= \int_0^l \frac{2x}{l} - \frac{2x^2}{l^2} dx \\
&= \left. \frac{x^2}{l} - \frac{2x^3}{3l^2} \right|_{x=0}^{l} \\
&= l - \frac{2l}{3} = \frac{l}{3}
\end{aligned}
$$

$$
\begin{aligned}
E[X^2] &= \int_0^l x^2 f_X(x) dx \\
&= \int_0^l \frac{2x^2}{l} - \frac{2x^3}{l^2} dx \\
&= \left. \frac{2x^3}{3l} - \frac{x^4}{2l^2} \right|_{x=0}^{l} \\
&= \frac{2}{3} l^2 - \frac{1}{2} l^2 = \frac{l}{6} l^2
\end{aligned}
$$

$$
var(X) = E[X^2] - E[X]^2 = \frac{1}{18} l^2
$$

For the second stick, the location of the second cut is dependent on the first, so we use the law of iterated expectation. Denote the location of the first cut C, which is uniform over $[0, l]$. Given C, Y is a mixed discrete-continuous random variable.

$$
Y = \begin{cases} C & w.p. 1/2 \\ unif([0, C]) & w.p. 1/2 \end{cases}
$$

Thus

$$
E[Y|C] = \frac{1}{2} C + \frac{1}{2} \frac{C}{2} = \frac{3}{4} C
$$

and

$$
E[Y] = E[E[Y|C]] = E\left[\frac{3}{4} C\right] = \frac{3}{4} E[C] = \frac{3}{8}.
$$

To find the variance, we use the law of total variance

$$
\begin{aligned}
var(Y) &= E[var(Y|C)] + var(E(Y|C)) \\
&= E\left[E[Y^2|C] - E[Y|C]^2\right] + var\left(\frac{3}{4} C\right) \\
&= E\left[\frac{1}{2} C^2 + \frac{1}{2} \frac{C^2}{3} - \left(\frac{3}{4} C\right)^2\right] + \frac{9}{16} var(C) \\
&= \frac{5}{48} E[C^2] + \frac{9}{16} \frac{l^2}{12} \\
&= \frac{5}{48} \frac{l^2}{3} + \frac{1}{64} l^2 \\
&= \frac{29}{154} l^2
\end{aligned}
$$