There are three ‘equivalent’ definitions of a jointly Gaussian (JG) random vector.

- A random vector \( X = (X_1, X_2, \ldots, X_k)^T \) is JG if there exists a base random vector \( Z = (Z_1, Z_2, \ldots, Z_\ell)^T \), each of which component is an independent standard normal random variable, a transition matrix \( A \in \mathbb{R}^{k \times \ell} \), and a mean vector \( \mu \in \mathbb{R}^k \), such that \( X = AZ + \mu \).

- A random vector \( X = (X_1, X_2, \ldots, X_k)^T \) is JG if \( Y = \sum_{i=1}^k a_i X_k \) is normally distributed for any \( a = (a_1, a_2, \ldots, a_k)^T \in \mathbb{R}^k \). Note: a point mass on a value is considered as a normal distribution with zero variance. Thus, if \( X_1 \sim N(0, 1) \) and \( X_2 = -X_1 \), \( X = (X_1, X_2)^T \) still is a JG random vector.

- (Degenerate case only) A random vector \( X = (X_1, X_2, \ldots, X_k)^T \) is JG if
  \[
  f_X(x) = \frac{1}{\sqrt{|\Sigma|(2\pi)^{n/2}}} \exp\left(-\frac{1}{2}(x - \mu)^T \Sigma^{-1}(x - \mu)\right).
  \]
  for a nonnegative-definite matrix \( \Sigma \in \mathbb{R}^{k \times k} \) and a mean vector \( \mu \in \mathbb{R}^k \).

Some useful facts:

- If \( (X, Y) \) is JG, \( \text{MMSE}[X|Y] = \text{LLSE}[X|Y] = E[X|Y] \).
- \( \text{MMSE}[X|Y] = \text{LLSE}[X|Y] \iff (X, Y) \) is JG.
- \( X_1 \) and \( X_2 \) are marginally distributed as Gaussian \( \iff X = (X_1, X_2)^T \) is JG.

**Problem 1.** (Fall 2015 Final) Let \( U, V \) be jointly Gaussian random variables with means \( \mu_U = 1, \mu_V = 4 \) and \( \sigma^2_U = 2.5, \sigma^2_V = 2 \) and covariance \( \rho = 1 \). Can we write \( U = aV + Z \) where \( a \) is a scalar and \( Z \) is independent of \( V \)? If so, find \( a \) and \( Z \), if not explain why.

**Problem 2.** (Fall 2015 Final) Assume that \( (X, Y_n, n \geq 0) \) are mutually independent random variables with \( X \sim N(0, 1) \), \( Y_n \sim N(0, 1) \). Let \( \hat{X}_n \) be the MMSE of \( X \) given \( X + Y_1, X + Y_2, \ldots, X + Y_n \).

(a) Show that \( \hat{X}_n = a_n(nX + \sum_{i=1}^n Y_i) \)

(b) Find \( a_n \).
Problem 3. (a) Consider zero-mean random variables $X, Y, Z$ such that $Y, Z$ are orthogonal. Show that $L[X|Y, Z] = L[X|Y] + L[X|Z]$.

(b) Show that for any random variables $X, Y, Z$ it holds that: