Problem 1. Midterm 01.

Problem 2. Consider a random variable $Z$ with transform:

$$M_Z(s) = \frac{a - 3s}{s^2 - 6s + 8}$$

(a) Find the numerical value for the parameter $a$.

(b) Find $P(Z \geq 0.5)$.

(c) Find $E[Z]$ by using the probability distribution of $Z$.

(d) Find $E[Z]$ by using the transform of $Z$ and without explicitly using the probability distribution of $Z$.

(e) Find $\text{Var}(Z)$ by using the probability distribution of $Z$.

(f) Find $\text{Var}(Z)$ by using the transform of $Z$ and without explicitely using the probability distribution of $Z$.

Problem 3. Let $X$, $Y$, and $Z$ be independent random variables. $X$ is Bernoulli with $p = 1/4$. $Y$ is exponential with parameter 3. $Z$ is Poisson with parameter 5.

(a) Find the transform of $5Z + 1$.

(b) Find the transform of $X + Y$.

(c) Consider the new random variable $U = XY + (1 - X)Z$. Find the transform associated with $U$.

Problem 4. In class, we learned some inequalities such as the Markov inequality, the Chebyshev inequality, and the Chernoff bound. In this problem, we will derive an inequality, which is a special case of Chernoff bound, using a simple counting method.

Suppose $X_1, \ldots, X_n$ are i.i.d. Bernoulli random variables with $\Pr(X_i = 1) = 1/2$. 
(a) First, use the Chebyshev inequality to show that for any $\epsilon > 0$,

$$\Pr \left( \sum_{i=1}^{n} X_i \geq \frac{n}{2} (1 + \epsilon) \right) \leq \frac{1}{\epsilon^2 n}. \tag{1}$$

The special case of Chernoff bound that we will derive is as follows: for any $\epsilon > 0$,

$$\Pr \left( \sum_{i=1}^{n} X_i \geq \frac{n}{2} (1 + \epsilon) \right) \leq \exp \{- \frac{\epsilon^2 n}{10}\}. \tag{2}$$

We will derive (2) in the next steps. We should notice that if $\epsilon > 1$, we have $\Pr(\sum_{i=1}^{n} X_i \geq \frac{n}{2} (1 + \epsilon)) = 0$. Therefore, we only need to consider the cases when $0 < \epsilon \leq 1$.

(b) Let $M$ be the event that $X_1 = X_2 = \cdots = X_m = 1, m < n$. Show that for an integer $k$ ($m \leq k \leq n$),

$$\Pr(M|\sum_{i=1}^{n} X_i = k) \geq \left( \frac{k-m}{n-m} \right)^m,$$

and further, show that

$$\Pr(M|\sum_{i=1}^{n} X_i \geq k) \geq \left( \frac{k-m}{n-m} \right)^m.$$

(c) For simplicity, we assume that $\frac{\epsilon n}{4}$ is an integer and let $m = \frac{\epsilon n}{4}$. Let $G$ be the event that $\sum_{i=1}^{n} X_i \geq \frac{n}{2} (1 + \epsilon)$. Show that

$$\Pr(M|G) \geq \left( \frac{1}{2} + \frac{\epsilon}{4} \right)^m.$$

(d) Show that $\Pr(M) \geq \Pr(G) \Pr(M|G)$. Then show that

$$\Pr(G) \leq \left( 1 + \frac{\epsilon}{2} \right)^{-m}.$$

(e) Combining the fact that for any $0 < \epsilon \leq 1$,

$$\ln(1 + \frac{\epsilon}{2}) > \frac{2}{5} \epsilon, \tag{3}$$

show that (2) holds. (You do not need to prove (3).)

(f) Compare (1) and (2) and argue why the Chernoff bound is better than the Chebyshev inequality.