1. Random Graph

Consider a random undirected graph on $n$ vertices, where each of the $\binom{n}{2}$ possible edges is present with probability $p$ independently of all the other edges. If $p = 0$ we have a fully empty graph with $n$ completely disconnected vertices; in contrast, if $p = 1$, every edge exists, the network is an $n$-clique, and every vertex is a distance one from every other vertex.

(a) Fix a particular vertex of the graph, and let $D$ be a random variable which is equal to the degree of this vertex. What is the PMF of $D$? Calculate $\lambda \triangleq \mathbb{E}[D]$.

(b) Assume that $c = np$ is a constant, independent of $n$. For large values of $n$, how would you approximate the PMF of $D$?

2. Isolated Vertices

Consider a Erdős-Renyi random graph $G(n, p(n))$, where $n$ is the number of vertices and $p(n)$ is the probability that a specific edge appears in the graph. Let $X_n$ be the number of isolated vertices in $G(n, p(n))$. Show that

$$\mathbb{E}[X_n] \xrightarrow{n \to \infty} \begin{cases} \infty, & p(n) \ll \frac{\ln n}{n}, \\ \exp(-c), & p(n) = \frac{\ln n + c}{n}, \\ 0, & p(n) \gg \frac{\ln n}{n}, \end{cases}$$
where the notation $p(n) \ll f(n)$ means that $p(n)/f(n) \to 0$ as $n \to \infty$, and $p(n) \gg f(n)$ means $p(n)/f(n) \to \infty$ as $n \to \infty$. Show also that in the third case, $p(n) \gg (\ln n)/n$, we have $X_n \to 0$ in probability as well.

3. Sub-Critical Forest

Consider a random graph $G(n, p(n))$ where $p(n) \ll 1/n$ (this is called the sub-critical phase). Show that the probability that $G(n, p(n))$ is a forest, i.e. contains no cycles, tends to 1 as $n \to \infty$. [If $X_n$ is the number of cycles, compute $\mathbb{E}[X_n]$ and show that $\mathbb{E}[X_n] \to 0$. Then, apply the First Moment Method.]