EECS 126: Probability & Random Processes

Midterm 2 Cheat Sheet

Spring 2019

1 Distributions

- $X \sim \text{Bernoulli}(p)$, $p \in [0,1]$.
  PMF: $p_X(x) = p^x(1-p)^{1-x}$, $x \in \{0,1\}$.
  MGF: $M_X(s) = 1 - p + p\exp s$.
  Moments: $\mathbb{E}[X] = p$, $\operatorname{var} X = p(1-p)$.

- $X \sim \text{Binomial}(n,p)$, $n \in \mathbb{Z}_+$, $p \in [0,1]$.
  PMF: $p_X(x) = \binom{n}{x}p^x(1-p)^{n-x}$, $x \in \{0,\ldots,n\}$.
  MGF: $M_X(s) = (1 - p + p\exp s)^n$.
  Moments: $\mathbb{E}[X] = np$, $\operatorname{var} X = np(1-p)$.

- $X \sim \text{Geometric}(p)$, $p \in (0,1)$.
  PMF: $p_X(x) = pq^{x-1}$, $x \in \mathbb{Z}_+$, $q = 1 - p$.
  MGF: $M_X(s) = (p\exp s)/(1 - q\exp s)$, $s < \ln(1/q)$.
  Moments: $\mathbb{E}[X] = p^{-1}$, $\operatorname{var} X = q/p^2$.

- $X \sim \text{Poisson}(\lambda)$, $\lambda > 0$.
  PMF: $p_X(x) = \lambda^x \exp(-\lambda)/x!$, $x \in \mathbb{N}$.
  MGF: $M_X(s) = \exp(\lambda(\exp s - 1))$.
  Moments: $\mathbb{E}[X] = \lambda$, $\operatorname{var} X = \lambda$.
  X, Y independent, $X \sim \text{Poisson}(\lambda)$, $Y \sim \text{Poisson}(\mu) \implies X + Y \sim \text{Poisson}(\lambda + \mu)$.

- $X \sim \text{Uniform}[a,b]$, $a < b$.
  PDF: $f_X(x) = (b-a)^{-1}$, $x \in [a,b]$.
  MGF: $M_X(s) = (\exp(sb) - \exp(sa))/(s(b-a))$.
  Moments: $\mathbb{E}[X] = (a+b)/2$, $\operatorname{var} X = (b-a)^2/12$.

- $X \sim \text{Exponential}(\lambda)$, $\lambda > 0$.
  PDF: $f_X(x) = \lambda \exp(-\lambda x)$, $x > 0$.
  CDF: $F_X(x) = (1 - \exp(-\lambda x))\mathbb{1}_{\{x \geq 0\}}$.
  MGF: $M_X(s) = \lambda/(\lambda - s)$, $s < \lambda$.
  Moments: $\mathbb{E}[X] = \lambda^{-1}$, $\operatorname{var} X = \lambda^{-2}$.

- $X \sim \mathcal{N}(\mu, \sigma^2)$, $\mu \in \mathbb{R}$, $\sigma^2 > 0$.
  PDF: $f_X(x) = (\sqrt{2\pi}\sigma)^{-1}\exp(-(x-\mu)^2/(2\sigma^2))$.
  CDF: $F_X(x) = \Phi(x)$.
  MGF: $M_X(s) = \exp(\mu s + \sigma^2 s^2/2)$.
  Moments: $\mathbb{E}[X] = \mu$, $\operatorname{var} X = \sigma^2$.
  X, Y independent, $X \sim \mathcal{N}(\mu_1, \sigma_1^2)$, $Y \sim \mathcal{N}(\mu_2, \sigma_2^2) \implies X + Y \sim \mathcal{N}(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$.

Continued:

- $X \sim \text{Pascal}(k,p)$, $k \in \mathbb{Z}_+$, $p \in (0,1)$.
  Sum of k i.i.d. Geometric(p).
  PMF: $p_X(x) = \binom{x-1}{k-1}p^k(1-p)^{x-k}$, $x = k, k+1, k+2, \ldots$.

- $X \sim \text{Erlang}(k,\lambda)$, $k \in \mathbb{Z}_+$, $\lambda > 0$.
  Sum of k i.i.d. Exponential(λ).
  PDF: $f_X(x) = \lambda^k x^{k-1}\exp(-\lambda x)/(k-1)!$, $x \geq 0$.

2 Definitions & Equations

<u>Tail Sum</u>: For $X \geq 0$, $\mathbb{E}[X] = \int_0^\infty \mathbb{P}(X \geq x)\,\mathrm{d}x$.

<u>Variance</u>: $\operatorname{var} X = \mathbb{E}[(X - \mathbb{E}[X])^2] = \mathbb{E}[X^2] - \mathbb{E}[X]^2$. <u>Sum</u>: $\operatorname{var}\sum_{i=1}^n X_i = \sum_{i=1}^n \operatorname{var} X_i + \sum_{i \neq j}\operatorname{cov}(X_i, X_j)$.

<u>Covariance</u>: $\operatorname{cov}(X,Y) = \mathbb{E}[XY] - \mathbb{E}[X]\,\mathbb{E}[Y]$. <u>Matrix</u>: If $X = (X_1, \ldots, X_n)$, $(\operatorname{cov} X)_{i,j} = \operatorname{cov}(X_i, X_j)$.

<u>Correlation</u>: $\rho(X,Y) = \operatorname{cov}(X,Y)/\sqrt{(\operatorname{var} X)(\operatorname{var} Y)}$.

<u>Law of total variance</u>: $\operatorname{var}(X) = \operatorname{var}(\mathbb{E}[X|Y]) + \mathbb{E}[\operatorname{var}(X|Y)]$

<u>Entropy</u>: $H(X) = -\sum_{x \in \mathcal{X}} p(x)\log_2 p(x) = -\mathbb{E}[\log_2 p(X)]$.

<u>Order Statistics</u>: $f_{X^{(i)}}(x) = n\binom{n-1}{i-1}f(x)F(x)^{i-1}(1-F(x))^{n-i}$.

<u>MGF</u>: $M_X(s) = \mathbb{E}[\exp(sX)]$.

<u>Markov</u>: For $X \geq 0$, $x > 0$, $\mathbb{P}(X \geq x) \leq \mathbb{E}[X]/x$.

<u>Chebyshev</u>: For $x > 0$, $\mathbb{P}(|X - \mathbb{E}[X]| \geq x) \leq (\operatorname{var} X)/x^2$.

<u>Chernoff</u>: For all x, $\mathbb{P}(X \geq x) \leq (M_X(s))/e^{sx}$ for all $s > 0$ where the MGF is defined.

<u>Geometric Series</u>: $\sum_{k=0}^{n-1} ar^k = \frac{a(1-r^n)}{1-r}, r \neq 1$