1. Colored Sphere

Consider a sphere that has $\frac{1}{10}$ of its surface colored blue, and the rest is colored red. Show that, no matter how the colors are distributed, it is possible to inscribe a cube in the sphere with all of its vertices red.

*Hint: Carefully define some relevant events.*

**Solution:**

Pick an inscribed cube uniformly at random, enumerate its vertices, and let $B_i$ be the event that vertex $i$ is blue. Note that:

$$
P(B_1 \cup \cdots \cup B_8) \leq \sum_{i=1}^{8} P(B_i) = \sum_{i=1}^{8} \frac{1}{10} = \frac{8}{10} < 1
$$

In other words, the probability of at least one vertex being blue is less than 1, so there must exist an inscribed cube where each vertex is red.

*Note:* This is an example of a powerful tool known as the probabilistic method.

2. Joint Occurrence

You know that, at least one of the events $A_r$ (for $r \in \{1, \ldots, n\}$, where $n$ is an integer $\geq 2$) is certain to occur but certainly no more than two occur. Show that if the probability of occurrence of any single event is $p$, and the probability of joint occurrence of any two distinct events is $q$, we have $p \geq 1/n$ and $q \leq 2/[n(n-1)]$.

**Solution:**

Since $1 = P(\bigcup_{r=1}^n A_r) \leq \sum_{r=1}^n P(A_r) = np$, we see that $p \geq 1/n$.

Let $I := \{(i, j) \in \{1, \ldots, n\}^2 : i < j\}$ be the set of pairs of distinct indices, avoiding repetition. Notice that the events $\{A_i \cap A_j : (i, j) \in I\}$ are pairwise disjoint, so by countable additivity,

$$
1 \geq P\left( \bigcup_{(i, j) \in I} (A_i \cap A_j) \right) = \sum_{(i, j) \in I} P(A_i \cap A_j) = \binom{n}{2} q,
$$

so $q \leq \binom{n}{2}^{-1} = 2/[n(n-1)]$.

3. Balls & Bins

Let $n \in \mathbb{Z}_{>1}$. You throw $n$ balls, one after the other, into $n$ bins, so that each ball lands in one of the bins uniformly at random. What is an appropriate
sample space to model this scenario? What is the probability that exactly one bin is empty?

**Solution:**

An appropriate sample space is to take $\Omega = \{1, \ldots, n\}^n$, the set of $n$-tuples where each coordinate is a number in $\{1, \ldots, n\}$. An outcome $\omega \in \Omega$ represents a scenario as follows: the first coordinate gives the label of the bin into which the first ball fell; the second coordinate gives the label of the bin into which the second ball fell; and so on.

Notice that this choice of sample space treats all of the balls as distinguishable and all of the bins as distinguishable. The reason for making this choice is that the sample space is uniform, that is, all outcomes have the same probability.

In contrast, if we chose a sample space corresponding to indistinguishable balls (and distinguishable bins), then the sample space would not be uniform, which makes the problem harder to analyze. The reason why the sample space is no longer uniform is that some outcomes can happen in more ways, so they have higher probabilities. Concretely, the outcome that all balls land in the first bin will have a smaller probability than the outcome that half the balls land in the first bin and the other half land in the second bin, because in the latter outcome you have the freedom to change which balls land in first bin (because the balls are indistinguishable).

Now, we return to our uniform sample space with distinguishable balls. The probability of each outcome is $n^{-n}$, so we must count how many outcomes correspond to exactly one empty bin. There are $n$ ways to choose which bin is empty; then $n - 1$ ways to choose which of the remaining bins will have two balls; then there are $\binom{n}{2}$ ways to choose which two of the $n$ balls will land in the bin with two balls; finally, there are $(n - 2)!$ ways to throw the remaining $n - 2$ balls into the $n - 2$ other bins. Therefore, the total number of outcomes is $n(n - 1)(n - 2)!\binom{n}{2} = n!\binom{n}{2}$, so the desired probability is $n!\binom{n}{2}/n^n$.

4. **[Bonus] Borel-Cantelli Lemma**

Prove the Borel-Cantelli Lemma: If $A_1, A_2, \ldots$ is a sequence of events with $\sum_{i=1}^{\infty} P(A_i) < \infty$, then

$$P(\text{infinitely many of } A_1, A_2, \ldots \text{ occur}) = 0.$$  

**Solution:**

If infinitely many of $A_1, A_2, \ldots$ occur, then at least one of $A_n, A_{n+1}, \ldots$ occurs for any $n \in \mathbb{Z}_{>0}$. So,

$$P(\text{infinitely many of } A_1, A_2, \ldots \text{ occur}) \leq P\left(\bigcup_{m=n}^{\infty} A_m\right) \leq \sum_{m=n}^{\infty} P(A_m) \xrightarrow{n \to \infty} 0$$

because $\sum_{i=1}^{\infty} P(A_i)$ converges. In more detail,

$$\sum_{m=n}^{\infty} P(A_m) = \sum_{m=1}^{\infty} P(A_m) - \sum_{m=1}^{n-1} P(A_m),$$
and as $n \to \infty$, the second term converges to $\sum_{m=1}^{\infty} \mathbb{P}(A_m)$, so $\sum_{m=n}^{\infty} \mathbb{P}(A_m)$ converges to 0 as $n \to \infty$.

Note: This result is incredibly useful for proving convergence results.