1. Generating Erdös-Renyi Random Graphs

*True/False:* Let $G_1$ and $G_2$ be independent Erdös-Renyi random graphs on $n$ vertices with probabilities $p_1$ and $p_2$, respectively. Let $G = G_1 \cup G_2$, that is, $G$ is generated by combining the edges from $G_1$ and $G_2$. Then, $G$ is an Erdös-Renyi random graph on $n$ vertices with probability $p_1 + p_2$.

2. Isolated Vertices

Consider an Erdös-Renyi random graph $G(n, p(n))$, where $n$ is the number of vertices and $p(n)$ is the probability that a specific edge appears in the graph. Let $X_n$ be the number of isolated vertices in $G(n, p(n))$. Show that

\[
\mathbb{E}[X_n] \xrightarrow{n \to \infty} \begin{cases} 
\infty, & p(n) \ll \frac{\ln n}{n}, \\
\exp(-c), & p(n) = \frac{\ln n + c}{n}, \\
0, & p(n) \gg \frac{\ln n}{n},
\end{cases}
\]

where the notation $p(n) \ll f(n)$ means that $p(n)/f(n) \to 0$ as $n \to \infty$, and $p(n) \gg f(n)$ means $p(n)/f(n) \to \infty$ as $n \to \infty$. Show also that in the third case, $p(n) \gg (\ln n)/n$, we have $X_n \to 0$ in probability as well.

3. Sub-Critical Forest

Consider a random graph $G(n, p(n))$ where $p(n) \ll 1/n$ (this is called the sub-critical phase). Show that the probability that $G(n, p(n))$ is a forest, i.e. contains no cycles, tends to 1 as $n \to \infty$. [If $X_n$ is the number of cycles, compute $\mathbb{E}[X_n]$ and show that $\mathbb{E}[X_n] \to 0$. Then, apply the First Moment Method.]