UC Berkeley
Department of Electrical Engineering and Computer Sciences

EECS 126: Probability and Random Processes

Problem Set 12
Spring 2019

Issued: Thursday April 25, 2019 **Due:** 11:59 PM, Friday May 3, 2019

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1. **Geometric MMSE**

   Let N be a geometric random variable with parameter $1-p$, and $(X_i)_{i\in\mathbb{N}}$ be i.i.d. exponential random variables with parameter λ. Let $T = X_1 + \cdots + X_N$. Compute the LLSE and MMSE of N given T.

2. **Property of MMSE**

   Let $X, Y_1, \ldots, Y_n$ be square integrable random variables. Argue that

$$\mathbb{E}\big[(X - \mathbb{E}[X \mid Y_1, \ldots, Y_n])^2\big] \le \mathbb{E}\left[\left(X - \sum_{i=1}^{n} \mathbb{E}[X \mid Y_i]\right)^2\right].$$

3. **Gaussian Random Vector MMSE**

   Let

$$\begin{bmatrix} X \\ Y \end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\right)$$

   be a Gaussian random vector.

   Let

$$W = \begin{cases} 1, & \text{if } Y > 0 \\ 0, & \text{if } Y = 0 \\ -1, & \text{if } Y < 0 \end{cases}$$

   be the sign of Y. Find $\mathbb{E}[WX \mid Y]$.

4. **Jointly Gaussian MMSE and Correlation Coefficients**

   (a) Provide justification for each of the following steps (1 - 5) to prove that the LLSE is equal to the MMSE estimator for jointly Gaussian random variables X and Y.

   Let $g(X) = L[Y|X]$.

$$\begin{aligned} &\mathbb{E}[(Y - g(X))X] = 0 && (1)\\ \implies &cov(Y - g(X), X) = 0 && (2)\\ \implies &Y - g(X) \text{ is independent of } X && (3)\\ \implies &\mathbb{E}[(Y - g(X))f(X)] = 0 \ \forall f(.) && (4)\\ \implies &g(X) = \mathbb{E}[Y|X] && (5) \end{aligned}$$

(b) Let X, Y, Z be jointly Gaussian random variables such that X is conditionally independent of Z given Y. Given the correlation coefficients of (X, Y) and (Y, Z) are ρ_1 and ρ_2, find the correlation coefficient of X and Z in terms of ρ_1 and ρ_2. For simplicity you may assume they are zero-mean, but the same answer holds even if they are not zero-mean.

(c) Let $X_0, X_1, ..., X_n$ be a jointly gaussian sequence that forms a Markov Chain such that the correlation coefficient of X_{i-1} and X_i is ρ_i. Find the correlation coefficient of X_0 and X_n in terms of $\rho_1, \rho_2, \ldots, \rho_n$ (Hint: use induction and part 2).

5. Stochastic Linear System MMSE

Let $(V_n,\ n \in \mathbb{N})$ be i.i.d. $\mathcal{N}(0, \sigma^2)$ and independent of $X_0 = \mathcal{N}(0, u^2)$. Let $|a| < 1$. Define

$$X_{n+1} = aX_n + V_n, \qquad n \in \mathbb{N}.$$

(a) What is the distribution of X_n, where n is a positive integer?

(b) Find $\mathbb{E}[X_{n+m} \mid X_n]$ for $m, n \in \mathbb{N}$, $m \geq 1$.

(c) Find u so that the distribution of X_n is the same for all $n \in \mathbb{N}$.

6. Random Walk with Unknown Drift

Consider a random walk with unknown drift. The dynamics are given, for $n \in \mathbb{N}$, as

$$\begin{aligned} X_1(n+1) &= X_1(n) + X_2(n) + V(n), \\ X_2(n+1) &= X_2(n), \\ Y(n) &= X_1(n) + W(n). \end{aligned}$$

Here, X_1 represents the position of the particle and X_2 represents the velocity of the particle (which is unknown but constant throughout time). Y is the observation. V and W are independent Gaussian noise variables with mean zero and variance σ_V^2 and σ_W^2 respectively.

(a) Write down the dynamics of the system in matrix-vector form and write down the Kalman filter recursive equations for this system.

(b) Let k be a positive integer. Compute the prediction $\mathbb{E}(X(n+k) \mid Y^{(n)})$, where $Y^{(n)}$ is the history of the observations $Y_0, \ldots, Y_n$, in terms of the estimate $\hat{X}(n) := \mathbb{E}(X(n) \mid Y^{(n)})$.

(c) Now let $k = 1$ and compute the smoothing estimate $\mathbb{E}(X(n) \mid Y^{(n+1)})$ in terms of the quantities that appear in the Kalman filter equation.

Hint: Use the law of total expectation

$$\mathbb{E}\big(X(n) \mid Y^{(n+1)}\big) = \mathbb{E}\big[\mathbb{E}\big(X(n) \mid X(n+1), Y^{(n+1)}\big) \mid Y^{(n+1)}\big],$$

as well as the *innovation*

$$\tilde{X}(n+1) := X(n+1) - L[X(n+1) \mid Y^{(n)}].$$

7. **[Bonus] Rotationally Invariant Random Variables**

The bonus question is just for fun. You are not required to submit the bonus question, but do give it a try and write down your progress.

You have two independent and identically distributed continuous random variables, with zero mean, such that the joint density is rotation invariant. Show that the random variables have the normal distribution.