Problem 1.  
1. Show that the probability that exactly one of the events $A$ and $B$ occurs is $P(A) + P(B) - 2P(A \cap B)$.

2. If $A$ is independent of itself, show that $P(A) = 0$ or $1$.

Solution 1.  
1. The probability of the event that exactly one of $A$ and $B$ occur is
\[ \Pr(A \cap B^c) + \Pr(A^c \cap B) = \Pr(A) - \Pr(A \cap B) + \Pr(B) - \Pr(A \cap B) = \Pr(A) + \Pr(B) - 2\Pr(A \cap B). \]

2. $\Pr(A \cap A) = \Pr(A) \Pr(A)$, so $\Pr(A) = \Pr(A)^2$; this implies that $\Pr(A) \in \{0, 1\}$.

Alternatively, suppose for the sake of contradiction that $0 < \Pr(A) < 1$. Then, $\Pr(A \mid A) = 1 \neq \Pr(A)$, which contradicts the supposed independence of $A$ with itself. Hence, $\Pr(A) \in \{0, 1\}$.

Problem 2. You know that, at least one of the events $A_r$ (for $r \in \{1, \ldots, n\}$, where $n$ is an integer $\geq 2$) is certain to occur but certainly no more than two occur. Show that if the probability of occurrence of any single event is $p$, and the probability of joint occurrence of any two distinct events is $q$, we have $p \geq 1/n$ and $q \leq 2/[n(n-1)]$.

Solution 2. Since $1 = \Pr(\bigcup_{r=1}^n A_r) \leq \sum_{r=1}^n \Pr(A_r) = np$, we see that $p \geq 1/n$.

Let $I := \{(i,j) \in \{1, \ldots, n\}^2 : i \neq j\}$ be the set of pairs of distinct indices, avoiding repetition. Notice that the events $\{A_i \cap A_j : (i,j) \in I\}$ are pairwise disjoint, so by countable additivity,
\[ 1 \geq \Pr\left( \bigcup_{(i,j) \in I} (A_i \cap A_j) \right) = \sum_{(i,j) \in I} \Pr(A_i \cap A_j) = \binom{n}{2} q, \]

so $q \leq \binom{n}{2}^{-1} = 2/[n(n-1)]$.

Problem 3. Consider a sphere that has $\frac{1}{10}$ of its surface colored blue, and the rest is colored red. Show that, no matter how the colors are distributed, it is possible to inscribe a cube in the sphere with all of its vertices red.

Hint: Carefully define some relevant events.
**Solution 3.** Pick an inscribed cube uniformly at random, enumerate its vertices, and let $B_i$ be the event that vertex $i$ is blue. Note that:

$$\Pr(B_1 \cup \cdots \cup B_8) \leq \sum_{i=1}^{8} \Pr(B_i) = \sum_{i=1}^{8} \frac{1}{10} = \frac{8}{10} < 1$$

In other words, the probability of at least one vertex being blue is less than 1, so there must exist an inscribed cube where each vertex is red.

**Note:** This is an example of a powerful tool known as the probabilistic method.

**Problem 4. [Extra] The Countable Union Bound**
Let $A_1, A_2, \ldots$ be a countable sequence of events. Prove that the union bound holds for countably many events:

$$\Pr\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} \Pr(A_i).$$

**Solution 4.** Define $A'_1 = A_1$ and $A'_i = A_i \setminus \bigcup_{j=1}^{i-1} A_j$ for $i \in \mathbb{N}, i \geq 2$. Now, the $A'_i$ for $i \in \mathbb{Z}_{>0}$ are disjoint, and $\bigcup_{i=1}^{\infty} A'_i = \bigcup_{i=1}^{\infty} A_i$, so $\Pr(\bigcup_{i=1}^{\infty} A_i) = \Pr(\bigcup_{i=1}^{\infty} A'_i) = \sum_{i=1}^{\infty} \Pr(A'_i)$. Also for all $i \in \mathbb{Z}_{>0}$ we have $\Pr(A'_i) \leq \Pr(A_i)$ since $A'_i \subseteq A_i$, so $\Pr(\bigcup_{i=1}^{\infty} A_i) \leq \sum_{i=1}^{\infty} \Pr(A_i)$.

Note: The fact we used above is that if $B \subseteq A$, then $\Pr(B) \leq \Pr(A)$; this follows because $A = B \cup (A \setminus B)$ is a disjoint union, so $\Pr(A) = \Pr(B) + \Pr(A \setminus B) \geq \Pr(B)$. 

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