Problem 1. Expected Norm
Pick two points $X$ and $Y$ independently and uniformly in $[0, 1]^2$. Calculate $E[∥X - Y∥_2^2]$.

Solution 1. If we let $X = (X_1, X_2)$ and $Y = (Y_1, Y_2)$, then $X_1, X_2, Y_1, Y_2 \sim Uniform[0, 1]$ and

$$E[∥X - Y∥_2^2] = E[(X_1 - Y_1)^2] + E[(X_2 - Y_2)^2].$$

We can calculate $E[(X_1 - Y_1)^2] = E[X_1^2] - 2E[X_1]E[Y_1] + E[Y_1^2] = 2/3 - 1/2 = 1/6.$

So, $E[∥X - Y∥_2^2] = 1/3.$

Problem 2. Generating Random Variables
Consider a continuous random variable $U \sim Uniform[0, 1]$. Let $F : \mathbb{R} \to [0, 1]$ be a strictly increasing distribution function. Show that $F^{-1}(U)$ has the cumulative distribution function (CDF) $F$.

Solution 2. Let $Y = F^{-1}(U)$. The CDF of $Y$ is $G(y) = P(Y \leq y) = P(U \leq F(y)) = F(y)$. The last equality follows from the CDF of a uniform random variable. Hence, $F^{-1}(U)$ has CDF $F$.

Problem 3. Derived Density
(a) Suppose $X \sim Uniform[-\pi/2, \pi/2]$. What is the density of $Y = tan(X)$.
(b) Show that the expectation of the derived density does not exist.

Solution 3. (a) Let $f(.)$ be the density of the random variable $X$, and $g(.)$ be the density of the random variable $Y$. We have,

$$P(y < Y < y + \delta) \approx g(y)\delta$$

for a small $\delta$. So, we have

$$P(y < Y < y + \delta) = P(y < tan(X) < y + \delta)$$

$$= P\left(\tan^{-1}(y) < Y < \tan^{-1}(y + \delta)\right)$$

$$\approx P\left(\tan^{-1}(y) < Y < \tan^{-1}(y) + \frac{d}{dy}\tan^{-1}y\right)$$
where the last line uses Taylor expansion. Continuing, we get

\[
P(y < Y < y + \delta) = P\left(\tan^{-1}(y) < Y < \tan^{-1}(y) + \delta \frac{1}{1+y^2}\right) \\
\approx f(\tan^{-1}(y)) \frac{\delta}{1+y^2}.
\]

Hence, we obtain,

\[
g(y) = f(\tan^{-1}(y)) \frac{1}{1+y^2}.
\]

Since, \(y = \tan^{-1}(x)\) and \(X \sim Uniform[-\pi/2, \pi/2]\), we get, \(f(\tan^{-1}(y)) = \frac{1}{\pi}\).

So the density of \(Y\) is,

\[
g(y) = \frac{1}{\pi(1+y^2)},
\]

with support \((-\infty, \infty)\). This density is known as Cauchy density, and comes up while analyzing the ratio of 2 Gaussians.

**Fun Fact:** The ratio of two independent Gaussian random variables is Cauchy. Can you prove it?

(b) Let us compute the expectation. We have

\[
\mathbb{E}(Y) = \int_{-\infty}^{\infty} y \frac{1}{\pi(1+y^2)} dy
\]

We now substitute, \(1 + y^2 = t\), and hence \(2ydy = dt\). We get,

\[
\mathbb{E}(Y) = \frac{1}{2\pi} \int_{\infty}^{\infty} \frac{1}{t} dt \\
= \frac{1}{2\pi} [\log t]_{\infty}^{\infty},
\]

which is of the form \(\infty - \infty\), and hence undefined.

**Fun Fact:** For the existence of expectation, we require a property called absolute integrability, which says that if \(\int |y|g(y)dy < \infty\), the expectation exists. One can easily check that for Cauchy random variables, the above integration is \(\infty\).