Problem 1. Linearity of Expectation, Independence
Let $X_1, X_2, X_3$ be discrete independent random variables with mean 0. Find $E[(X_1 + X_2)(X_2 + X_3)(X_3 + X_1)]$.

Solution 1. $E[(X_1 + X_2)(X_2 + X_3)(X_3 + X_1)]$ contains terms of the form $E(X_i^2X_j)$, $i \neq j$ and $E(X_iX_jX_k)$, $i \neq j \neq k$. Since $X_1, X_2, X_3$ are independent and zero mean, $E(X_i^2X_j) = E(X_i^2)E(X_j) = 0$, and from the same logic, $E(X_iX_jX_k) = 0$. Hence, $E[(X_1 + X_2)(X_2 + X_3)(X_3 + X_1)] = 0$.

Problem 2. Drawing Batteries
You have an endless box of used batteries. Assume that the number of hours remaining in a battery is i.i.d., uniformly distributed on $[0, 1]$.

1. Let $X$ and $Y$ denote the lifetimes of the first and second batteries you draw. What is $\Pr(X^{1/2} > Y)$?


Solution 2. 1. One has
$$\Pr(X^{1/2} > Y) = E[\Pr(X^{1/2} > Y \mid Y)] = E[\Pr(X > Y^2 \mid Y)]$$
$$= E[1 - Y^2] = 1 - \frac{1}{3} = \frac{2}{3}.$$

2. No. Observe that
$$\Pr\left(X + Y < \frac{1}{2}, X - Y > \frac{1}{2}\right) = 0,$$
since $X + Y < 1/2$ implies that $X < 1/2$ so that $X - Y > 1/2$ is impossible. However, clearly $\Pr(X + Y < 1/2) > 0$ and $\Pr(X - Y > 1/2) > 0$.

Problem 3. Convergence in Probability
Let $(X_n)_{n=1}^\infty$, be a sequence of i.i.d. random variables distributed uniformly in $[-1, 1]$. Show that the following sequences $(Y_n)_{n=1}^\infty$ converge in probability to some limit.

(a) $Y_n = \prod_{i=1}^n X_i$.

(b) $Y_n = \max\{X_1, X_2, \ldots, X_n\}$. 
(c) $Y_n = (X_1^2 + \cdots + X_n^2)/n$.

**Solution 3.**

(a) By independence of the random variables,

$$
E[Y_n] = E[X_1] \cdots E[X_n] = 0,
$$

$$
varY_n = E[Y_n^2] = (varX_1)^n = \left(\frac{1}{3}\right)^n.
$$

Now since $varY_n \to 0$ as $n \to \infty$, by Chebyshev’s Inequality the sequence converges to its mean, that is, 0, in probability.

(b) Consider $\epsilon \in [0,1]$. We see that:

$$
Pr(|Y_n - 1| \geq \epsilon) = Pr(\max\{X_1,\ldots,X_n\} \leq 1 - \epsilon)
$$

$$
= Pr(X_1 \leq 1 - \epsilon,\ldots,X_n \leq 1 - \epsilon)
$$

$$
= Pr(X_1 \leq 1 - \epsilon)^n = \left(1 - \frac{\epsilon}{2}\right)^n
$$

Thus, $Pr(|Y_n - 1| \geq \epsilon) \to 0$ as $n \to \infty$ and we are done.

(c) The expectation is

$$
E[Y_n] = \frac{1}{n} \cdot nE[X_1^2] = \frac{1}{3}.
$$

Then, we bound the variance.

$$
VarY_n = \frac{1}{n} VarX_1^2 \leq \frac{1}{n} \to 0 \quad \text{as } n \to \infty,
$$

since $X_1^2 \leq 1$. Hence, we see that $Y_n \to 1/3$ in probability as $n \to \infty$.

**Remark:** We now provide an interpretation for the previous result. The sample space for $Y_n$ is $\Omega_n = [-1,1]^n$, which is an $n$-dimensional cube. The result we have just proved shows that, for any $\epsilon > 0$, the set

$$
B_n = \left\{ x \in \mathbb{R}^n : \frac{1}{3}(1 - \epsilon) \leq \frac{x_1^2 + \cdots + x_n^2}{n} \leq \frac{1}{3}(1 + \epsilon) \right\}
$$

makes up “most” of the volume of $\Omega_n$, in the sense that

$$
\frac{\text{volume}(B_n \cap [-1,1]^n)}{2^n} \to 1 \quad \text{as } n \to \infty.
$$

Since $B_n$ is close to the boundary of a ball of radius $\sqrt{n/3}$, the result can be stated facetiously as “nearly all of the volume of a high-dimensional cube is contained in the boundary of a ball”. Although this may seem like a meaningless comment, in fact various phenomena such as these contribute to the so-called “curse of dimensionality” in machine learning, which concerns the sparsity of data in high-dimensional statistics.