# 1. Random Walk on an Undirected Graph

Consider a random walk on an undirected connected finite graph (that is, define a Markov chain where the state space is the set of vertices of the graph, and at each time step, transition to a vertex chosen uniformly at random out of the neighborhood of the current vertex). What is the stationary distribution?

**Solution:**

Let $\mathcal{X}$ be the state space. The stationary distribution is

$$\pi(v) = \frac{\deg(v)}{\sum_{v' \in \mathcal{X}} \deg(v')}, \quad v \in \mathcal{X}.$$  

Clearly, $\pi$ is a valid probability distribution. We check that the chain is reversible. Note that if $u$ and $v$ are neighbors, then

$$\pi(u)P(u,v) = \frac{\deg(u)}{\sum_{v' \in \mathcal{X}} \deg(v')} \cdot \frac{1}{\deg(u)} = \frac{1}{\sum_{v' \in \mathcal{X}} \deg(v')}.$$  

Also,

$$\pi(v)P(v,u) = \frac{\deg(v)}{\sum_{v' \in \mathcal{X}} \deg(v')} \cdot \frac{1}{\deg(v)} = \frac{1}{\sum_{v' \in \mathcal{X}} \deg(v')}.$$  

So, $\pi(u)P(u,v) = \pi(v)P(v,u)$ if $u$ and $v$ are neighbors. If $u$ and $v$ are not neighbors, then $P(u,v) = P(v,u) = 0$, so the equation holds in this case as well. The chain is reversible and so $\pi$ is stationary.

# 2. Reducible Markov Chain

Consider the following Markov chain, for $\alpha, \beta, p, q \in (0,1)$.

![Markov Chain Diagram]

(a) Find all the recurrent and transient classes.  

(b) Given that we start in state 2, what is the probability that we will reach state 0 before state 5?  

(c) What are all of the possible stationary distributions of this chain?
Suppose we start in the initial distribution  
\[ \pi_0 := \begin{bmatrix} 0 & 0 & \gamma & 1-\gamma & 0 & 0 \end{bmatrix} \]
for some  \( \gamma \in [0,1] \). Does the distribution of the chain converge, and if so, to what?

**Solution:**

(a) The classes are \{0, 1\} (recurrent), \{4, 5\} (recurrent), and \{2, 3\} (transient).

(b) Let  \( T_0 \) and  \( T_5 \) denote the time it takes to reach states 0 and 5 respectively. (Note that exactly one of  \( T_0 \) and  \( T_5 \) will be finite.) We are looking to compute  \( \mathbb{P}_2(T_0 < T_5) \), and we can set up hitting equations:

\[
\mathbb{P}_2(T_0 < T_5) = \frac{1}{2} + \frac{1}{2} \mathbb{P}_3(T_0 < T_5),
\]

\[
\mathbb{P}_3(T_0 < T_5) = \frac{1}{2} \mathbb{P}_2(T_0 < T_5).
\]

Thus,  \( \mathbb{P}_2(T_0 < T_5) = 2/3 \).

(c) First we observe that no stationary distribution can put positive probability on a transient state, so the stationary distribution is supported on the states \{0, 1, 4, 5\}. Next, if we restrict our attention to only the states \{0, 1\}, then we have an irreducible Markov chain with stationary distribution

\[ \pi_1 := \frac{1}{\alpha + \beta} \begin{bmatrix} \beta & \alpha \end{bmatrix}, \]

and similarly, if we restrict our attention to only the states \{4, 5\}, then again we have an irreducible Markov chain with stationary distribution

\[ \pi_2 := \frac{1}{p + q} \begin{bmatrix} q & p \end{bmatrix}. \]

Any stationary distribution for the entire chain must be some convex combination of these two stationary distributions. Explicitly, the stationary distributions are of the form

\[ \pi = \begin{bmatrix} \frac{c\beta}{\alpha + \beta} & \frac{c\alpha}{\alpha + \beta} & 0 & 0 & \frac{(1-c)q}{p+q} & \frac{(1-c)p}{p+q} \end{bmatrix} \tag{1} \]

for some  \( c \in [0,1] \).

(d) Indeed the distribution will converge, even though we do not have irreducibility. The intuition is as follows. The probability will leak out of the transient states \{2, 3\} until all of the probability mass is supported on the recurrent states. The two recurrent classes can each be considered to be an irreducible aperiodic Markov chain and so the probability mass which enters a recurrent class will settle into equilibrium. To aid us in finding the limiting distribution, we can use the results of Part (b). With probability  \( \gamma \), we start in state 2, and with a further probability 2/3 we end up in the recurrent class \{0, 1\}. By symmetry, the probability that we end up in \{0, 1\} starting form state 3 is 1/3. Thus, the total probability mass which settles into the recurrent class \{0, 1\} is 2\( \gamma \)/3 + (1-\( \gamma \))/3 = 1/3 + \( \gamma \)/3. Then,
the probability mass settling in the recurrent class \( \{4, 5\} \) is \( 2/3 - \gamma/3 \). Therefore, the chain converges to the stationary distribution in (1) with \( c = 1/3 + \gamma/3 \).

3. Customers in a Store

Consider two independent Poisson processes with rates \( \lambda_1 \) and \( \lambda_2 \). Those processes measure the number of customers arriving in store 1 and 2.

(a) What is the probability that a customer arrives in store 1 before any arrives in store 2?

(b) What is the probability that in the first hour exactly 6 customers arrive, in total, at the two stores?

(c) Given that exactly 6 have arrived, in total, at the two stores, what is the probability that exactly 4 went to store 1?

Solution:

(a) **Solution 1:** Consider the sum of two processes which is a Poisson process with rate \( \lambda_1 + \lambda_2 \). You mark each customer in this process as 1 with probability \( \lambda_1/(\lambda_1 + \lambda_2) \) and mark as 2 otherwise. The resulting two processes are Poisson processes of rates \( \lambda_1 \) and \( \lambda_2 \). Thus, the probability of having the first customer going to store 1 is equal to the probability of marking the first customer as 1 which is

\[
\frac{\lambda_1}{\lambda_1 + \lambda_2}.
\]

**Solution 2:** The arrival times of the first customer of the two stores are \( X \sim \text{Exponential}(\lambda_1) \) and \( Y \sim \text{Exponential}(\lambda_2) \), respectively. Then using the total probability theorem we have that

\[
\mathbb{P}(X < Y) = \int_0^\infty f_Y(y)\mathbb{P}(X < Y \mid Y = y) \, dy
\]

\[
= \int_0^\infty \lambda_2 e^{-\lambda_2 y} (1 - e^{-\lambda_1 y}) \, dy
\]

\[
= \frac{\lambda_1}{\lambda_1 + \lambda_2}.
\]

(b) \( e^{-\lambda_1 \lambda_2} \lambda_1 \lambda_2^5 / 6! \).

(c) \( \binom{6}{4} \left( \frac{\lambda_1}{\lambda_1 + \lambda_2} \right)^4 \left( \frac{\lambda_2}{\lambda_1 + \lambda_2} \right)^2 \).