1. Frogs
Three frogs are playing near a pond. When they are in the sun they get too hot and jump in the lake at rate 1. When they are in the lake they get too cold and jump onto the land at rate 2. The rates here refer to the rate in exponential distribution. Let $X_t$ be the number of frogs in the sun at time $t \geq 0$.

(a) Find the stationary distribution for $(X_t)_{t \geq 0}$.
(b) Check the answer to (a) by noting that the three frogs are independent two-state Markov chains.

2. Taxi Queue
Empty taxis pass by a street corner at a Poisson rate of two per minute and pick up a passenger if one is waiting there. Passengers arrive at the street corner at a Poisson rate of one per minute and wait for a taxi only if there are less than four persons waiting; otherwise they leave and never return. John arrives at the street corner at a given time. Find his expected waiting time, given that he joins the queue. Assume that the process is in steady state.

3. Poisson Queues
A continuous-time queue has Poisson arrivals with rate $\lambda$, and it is equipped with infinitely many servers. The servers can work in parallel on multiple customers, but they are non-cooperative in the sense that a single customer can only be served by one server. Thus, when there are $k$ customers in the queue ($k \in \mathbb{N}$), $k$ servers are active. Suppose that the service time of each customer is exponentially distributed with rate $\mu$ and they are i.i.d.

(a) Argue that the queue-length is a Markov chain. Draw the transition diagram of the Markov chain.
(b) Prove that for all finite values of $\lambda$ and $\mu$ the Markov chain is positive-recurrent and find the invariant distribution.