1. **Transform Practice**

Consider a random variable $Z$ with transform

$$M_Z(s) = \frac{a - 3s}{s^2 - 6s + 8}, \quad \text{for } |s| < 2.$$ 

Calculate the following quantities:

(a) The numerical value of the parameter $a$.

(b) $\mathbb{E}[Z]$.

(c) $\text{var}(Z)$.

2. **Combining Transforms**

Let $X$, $Y$, and $Z$ be independent random variables. $X$ is Bernoulli with $p = 1/4$. $Y$ is exponential with parameter 3. $Z$ is Poisson with parameter 5.

(a) Find the transform of $5Z + 1$.

(b) Find the transform of $X + Y$.

(c) Consider the new random variable $U = XY + (1 - X)Z$. Find the transform associated with $U$.

3. **Gaussian Densities**

(a) Let $X_1 \sim \mathcal{N}(0, 1)$, $X_2 \sim \mathcal{N}(0, 1)$, where $X_1$ and $X_2$ are independent. Convolve the densities of $X_1$ and $X_2$ to show that $X_1 + X_2 \sim \mathcal{N}(0, 2)$.

(b) Let $X \sim \mathcal{N}(0, \sigma^2)$; find $\mathbb{E}[X^n]$ for $n \in \mathbb{N}$.

(c) Let $Z \sim \mathcal{N}(0, 1)$. For a random vector $(X_1, \ldots, X_n)$ where $n$ is a positive integer and $X_1, \ldots, X_n$ are real-valued random variables, the expectation of $(X_1, \ldots, X_n)$ is the vector of elementwise expectations of each random variable and the **covariance matrix** of
$(X_1, \ldots, X_n)$ is the $n \times n$ matrix whose $(i, j)$ entry is $\text{cov}(X_i, X_j)$ for all $i, j \in \{1, \ldots, n\}$. Find the mean and covariance matrix of $(Z, 1\{Z > c\})$ in terms of $\phi$ and $\Phi$, the standard Gaussian PDF and CDF respectively.

4. A Chernoff Bound for the Sum of Coin Flips
Let $X_1, \ldots, X_n$ be i.i.d. Bernoulli($q$) random variables with bias $q \in (0, 1)$, and call $X$ their sum, $X = X_1 + \cdots + X_n$, which a Binomial($n, q$) random variable, with mean $\mathbb{E}[X] = nq$.

(a) Let $\epsilon > 0$ such that $q + \epsilon < 1$, and define $p = q + \epsilon$. Show that for any $t > 0$,
\[
P(X \geq pn) \leq \exp\left(-n(tp - \ln\mathbb{E}[e^{tX_1}])\right).
\]

(b) The Kullback-Leibler divergence from the distribution Bernoulli($q$) to the distribution Bernoulli($p$), is defined as
\[
D(p \parallel q) \triangleq p \ln \frac{p}{q} + (1 - p) \ln \frac{1 - p}{1 - q}.
\]

The Kullback-Leibler divergence can be interpreted as a measure of how close the two distributions are. One motivation for this interpretation is that the Kullback-Leibler divergence is always nonnegative, i.e. $D(p \parallel q) \geq 0$, and $D(p \parallel q) = 0$ if and only if $p = q$. So it can be thought of as a ‘distance’ between the two Bernoulli distributions.

Optimize the previous bound over $t > 0$ and deduce that
\[
P(X \geq pn) \leq e^{-nD(p \parallel q)}.
\]

(c) Moreover, the Kullback-Leibler divergence is related to the square distance between the parameters $p$ and $q$ via the following inequality
\[
D(p \parallel q) \geq 2(p - q)^2, \quad \text{for } p, q \in (0, 1).
\]

Use this inequality in order to deduce that
\[
P\left(X \geq (q + \epsilon)n\right) \leq e^{-2n\epsilon^2},
\]
and
\[
P\left(X \leq (q - \epsilon)n\right) \leq e^{-2n\epsilon^2}.
\]

*Hint:* For the second bound use symmetry in order to avoid doing all the work again.
(d) Conclude that

\[ P(|X - qn| \geq \epsilon n) \leq 2e^{-2n\epsilon^2}. \]