LECTURE 20

Optimality Conditions

Duality, in mathematics, principle whereby one true statement can be obtained from another by merely interchanging two words.

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Overview

In this lecture, we describe the so-called “optimality conditions” that characterize optimality for convex programs, and generalize the “zero-gradient” condition that arises in convex unconstrained problems.

These conditions have many uses, in particular in
- the theoretical analysis of solutions to convex problems;
- the design of convex optimization algorithms.

We will first look at an “abstract” form of optimality conditions that offer geometric insight and work well for equality constraints only; then develop optimality conditions for the general case.
Primal problem

In this lecture, we consider the following “primal” problem

\[ p^* = \min_{x \in \mathbb{R}^n} f_0(x) \text{ subject to: } f_i(x) \leq 0, \quad i = 1, \ldots, m, \]
\[ Ax = b, \]

where

- \( f_0, \ldots, f_m \) are convex differentiable functions, which we assume to be defined everywhere (hence the domain of the problem is \( D = \mathbb{R}^n \));
- matrix \( A \in \mathbb{R}^{q \times n} \) and vector \( b \in \mathbb{R}^q \) are given.

We denote by \( D \) the domain of the problem: \( D \doteq \bigcap_{i=0}^{m} \text{dom } f_i \).

We make a few assumptions on the above problem:

- it is strictly feasible (so that Slater’s condition holds);
- it is attained: there exist \( x^* \in D \) such that \( p^* = f_0(x^*) \).
Abstract form of optimality conditions

The primal problem can be written in abstract form

\[
\min_{x \in \mathcal{X}} f_0(x),
\]

where \( \mathcal{X} \subseteq \mathcal{D} \) denotes the feasible set.

**Proposition 1**

Consider the optimization problem \( \min_{x \in \mathcal{X}} f_0(x) \), where \( f_0 \) is convex and differentiable, and \( \mathcal{X} \) is convex. Then,

\[
x \in \mathcal{X} \text{ is optimal} \iff \nabla f_0(x)^\top (y - x) \geq 0, \quad \forall y \in \mathcal{X}.
\]  

(1)

**Note:** the above conditions are often hard to work with, due to the presence of the \( \forall y \ldots \) statement, which requires checking a condition over the entire feasible set.
Proof

First let us show the implication from right to left in (1). Since $f_0$ is convex, for every $x, y \in \text{dom } f_0$, we have

$$f_0(y) \geq f_0(x) + \nabla f_0(x)^\top (y - x).$$

(2)

The implication from right to left in (1) is immediate, since

$$\nabla f_0(x)^\top (y - x) \geq 0 \text{ for every } y \in \mathcal{X}$$

implies, from (2), that $f_0(y) \geq f_0(x)$ for all $y \in \mathcal{X}$, i.e., that $x$ is optimal.

Conversely, assume that $x$ is optimal. We show that then $\nabla f_0(x)^\top (y - x) \geq 0$ for all $y \in \mathcal{X}$. If $\nabla f_0(x) = 0$, then the claim holds trivially. Assume now that $\nabla f_0(x) \neq 0$, and that there exist $y \in \mathcal{X}$ such that $\nabla f_0(x)^\top (y - x) < 0$. Consider the function

$$g : t \in [0, 1] \rightarrow f_0(x(t)),$$

where $x(t) = ty + (1 - t)x$; note that $x(t) \in \mathcal{X}$ for every $t \in [0, 1]$, since $\mathcal{X}$ is convex. Further, $g'(0) = \nabla f_0(x)^\top (y - x)$. Hence, for sufficiently small $t > 0$, $g(t) < g(0)$, which translates as $f(x(t)) < f(x)$; with $x(t) \in \mathcal{X}$, this contradicts the optimality of $x$. □
Optimality conditions

Geometric interpretation

If $\nabla f_0(x) \neq 0$, then $\nabla f_0(x)$ is a normal direction defining an hyperplane
\[
\{ y : \nabla f_0(x)^\top (y - x) = 0 \}
\]
such that:

- $x$ is on the boundary of the feasible set $\mathcal{X}$, and
- the whole feasible set lies on one side of this hyperplane, that is in the halfspace
defined by
\[
\mathcal{H}_+(x) = \{ y : \nabla f_0(x)^\top (y - x) \geq 0 \}.
\]
Optimality conditions

Geometric interpretation

Notice that the gradient vector $\nabla f_0(x)$ defines two set of directions:

- for directions $v_+$ such that $\nabla f_0(x)^\top v_+ > 0$ (i.e., directions that have positive inner product with the gradient), if we make a move away from $x$ in direction $v_+$, then the objective $f_0$ increases.

- for directions $v_-$ such that $\nabla f_0(x)^\top v_- < 0$ (i.e., descent directions, that have negative inner product with the gradient), if we make a sufficiently small move away from $x$ in direction $v_-$, then the objective $f_0$ locally decreases.

Condition (1) then says that $x$ is an optimal point if and only if there is no feasible direction along which we may improve (decrease) the objective.
Proposition 2

In a convex unconstrained problem with differentiable objective, $x$ is optimal if and only if

$$\nabla f_0(x) = 0.$$  \hspace{1cm} (3)

Proof: When the problem is unconstrained, i.e., $\mathcal{X} = \mathbb{R}^n$, then the optimality condition (1) requires that

$$\forall y \in \mathbb{R}^n : \nabla f_0(x)^\top (y - x) \geq 0 \iff \forall z \in \mathbb{R}^n : \nabla f_0(x)^\top z \geq 0$$

$$\iff \forall z \in \mathbb{R}^n : \nabla f_0(x)^\top z = 0$$

$$\iff \nabla f_0(x) = 0.$$
Consider the problem
\[
\min_x f_0(x) : Ax = b,
\]
where \( A \in \mathbb{R}^{m \times n}, \ b \in \mathbb{R}^m \) are given. We assume that \( b \in \mathcal{R}(A) \), so the problem is feasible. Here the feasible set is
\[
\mathcal{X} = \{ y : Ay = b \}.
\]

**Proposition 3**

A point \( x \) is optimal for problem (4) if and only if
\[
Ax = b \text{ and } \exists \nu \in \mathbb{R}^m : \nabla f_0(x) + A^\top \nu = 0.
\]
Proof

The point $x \in \mathcal{X}$ is optimal iff

$$\nabla f_0(x)^\top (y - x) \geq 0, \quad \forall y \in \mathcal{X}.$$ 

Since $Ax = b$, the feasible set can be written as

$$\mathcal{X} = \{x + z : z \in \mathcal{N}(A)\}.$$

The optimality condition becomes

$$\forall z \in \mathcal{N}(A) : \nabla f_0(x)^\top z \geq 0.$$ 

Since $z \in \mathcal{N}(A)$ if and only if $-z \in \mathcal{N}(A)$, we see that the condition is equivalent to

$$\forall z \in \mathcal{N}(A) : \nabla f_0(x)^\top z = 0.$$ 

That is, $\nabla f_0(x) \in \mathcal{N}(A)^\perp$. Recall the fundamental theorem of linear algebra, which states that $\mathcal{N}(A)^\perp = \mathcal{R}(A^\top)$; we obtain that there exist $\nu \in \mathbb{R}^m$ such that

$$\nabla f_0(x) + A^\top \nu = 0.$$
Example

Minimum-norm solutions to linear equations

Consider the Euclidean projection problem seen in lecture 8:

$$\min_x \frac{1}{2} x^\top x : A x = b.$$  

(The solution is the projection of 0 on the affine subspace $\mathcal{X}$.)

We obtain that $x$ is optimal if and only if there exist $\nu \in \mathbb{R}^m$ such that

$$A x = b, \quad x + A^\top \nu = 0.$$  

(5)

Assuming that $A$ is full row rank (hence, $AA^\top \succ 0$), we get the unique solution:

$$\nu^* = -(AA^\top)^{-1} b, \quad x^* = -A^\top \nu^* = A^\top (AA^\top)^{-1} b.$$
General case

Dual problem

Turning to the general problem (1), recall the expression of the problem dual to (1), as seen in lecture 18:

$$d^* = \max_{\lambda \geq 0} g(\lambda),$$

(6)

where $g$ is the dual function

$$g(\lambda) = \min_x \mathcal{L}(x, \lambda, \nu),$$

with $\mathcal{L}$ the Lagrangian

$$\mathcal{L}(x, \lambda) = f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x).$$

- Since Slater’s condition hold, we have strong duality: $p^* = d^*$.
- We make the further assumption that $d^*$ is attained by some $\lambda^* \geq 0$. 
Karush-Kuhn-Tucker (KKT) conditions

For the convex problem (1), we say that a pair \((x, \lambda) \in \mathbb{R}^n \times \mathbb{R}^m\) satisfies the Karush-Kuhn-Tucker (KKT) conditions if

1. Primal feasibility: \(x\) is feasible for the primal problem:
   \[ x \in \mathcal{D}, \quad f_i(x) \leq 0, \quad i = 1, \ldots, m. \]

2. Dual feasibility: \(\lambda \geq 0\).

3. Complementary slackness: \(\lambda_i f_i(x) = 0, \quad i = 1, \ldots, m\).

4. Lagrangian stationarity: \(x \in \arg \min \mathcal{L}(\cdot, \lambda)\), which, in the case when the functions \(f_i, \ i = 0, \ldots, m\) are differentiable, writes
   \[ \nabla_x f_0(x) + \sum_{i=1}^m \lambda_i \nabla_x f_i(x) = 0. \]

Proposition 4

Assume that the primal problem (1) is convex, and attained; that its dual is also attained; and that strong duality holds. Then, a primal-dual pair \((x, \lambda)\) is optimal if and only if it satisfies the KKT conditions.
Proof: sufficiency

Assume that the KKT conditions are satisfied for some pair \((x^*, \lambda^*)\). The first two conditions imply that \(x^*\) is primal feasible, and \(\lambda^*\) is dual feasible. Further, since \(\mathcal{L}(x, \lambda^*)\) is convex in \(x\), the fourth condition states that \(x^*\) is a global minimizer of \(\mathcal{L}(x, \lambda^*)\), hence

\[
g(\lambda^*, \nu^*) = \min_{x \in D} \mathcal{L}(x, \lambda^*) = \mathcal{L}(x^*, \lambda^*)
\]

\[
= f_0(x^*) + \sum_{i=1}^{m} \lambda_i^* f_i(x^*)
\]

\[
= f_0(x^*),
\]

where the last equality follows from complementary slackness.

The above proves that the primal-dual feasible pair \((x^*, \lambda^*)\) is optimal: the corresponding duality gap \(p^* - d^*\) is zero, since \(x^*\) (resp. \(\lambda^*\)) attains the lower bound \(d^*\) (resp. upper bound \(p^*\)).
Proof: necessity

Assume that \((x^*, \lambda^*)\) is an optimal primal-dual pair.

- Since \(p^* = f_0(x^*), d^* = g(\lambda^*), \) and \(p^* = d^*,\) we have

\[
f_0(x^*) = g(\lambda^*) = \inf_{x \in \mathcal{D}} \mathcal{L}(x, \lambda^*) \leq \mathcal{L}(x, \lambda^*), \quad \forall x \in \mathcal{D}.
\]

- Since the last inequality holds for all \(x \in \mathcal{D},\) it must hold also for \(x^*,\) hence

\[
f_0(x^*) = \inf_{x \in \mathcal{D}} \mathcal{L}(x, \lambda^*) \leq \mathcal{L}(x^*, \lambda^*) = f_0(x^*) + \sum_{i=1}^{m} \lambda^*_i f_i(x^*) \leq f_0(x^*),
\]

where the last inequality follows from the fact that \(x^*\) is optimal, hence feasible, for the primal problem, therefore \(f_i(x^*) \leq 0,\) and \(\lambda^*\) is optimal, hence feasible, for the dual, therefore \(\lambda^*_i \geq 0,\) whereby each term \(\lambda^*_i f_i(x^*)\) is \(\leq 0.\)

- Observing the last chain of inequalities, since the first and the last terms are equal, we must conclude that all inequalities must actually hold with equality, that is

\[
f_0(x^*) = \inf_{x \in \mathcal{D}} \mathcal{L}(x, \lambda^*) = \mathcal{L}(x^*, \lambda^*).
\]
Complementary slackness and Lagrangian stationarity

These two conditions are at the heart of the KKT conditions.

The complementary slackness property prescribes that a primal and the corresponding dual inequality cannot be slack simultaneously, that is, if $f_i(x^*) < 0$, then it must be $\lambda^*_i = 0$, and if $\lambda^*_i > 0$, then it must be $f_i(x^*) = 0$.

The second property (i.e., the fact that $x^*$ is a minimizer of $\mathcal{L}(x, \lambda^*)$) can, in some cases, be used to recover a primal-optimal variable from the dual-optimal variables (see later).
Recovering primal solutions from the dual

- First observe that if the primal problem is convex, then \( L(x, \lambda^*) \) is also convex in \( x \). Global minimizers of this function can then be determined by unconstrained minimization techniques. For instance, if \( L(x, \lambda^*) \) is differentiable, a necessary condition for \( x \) to be a global minimizer is determined by the zero-gradient condition \( \nabla_x L(x, \lambda^*) = 0 \), that is,

\[
\nabla_x f_0(x) + \sum_{i=1}^{m} \lambda_i^* \nabla_x f_i(x) = 0.
\]

- However, \( L(x, \lambda^*) \) may have multiple global minimizers, and it is not guaranteed that every global minimizer of \( L \) is a primal-optimal solution—what is guaranteed is that the primal-optimal solution \( x^* \) is among the global minimizers of \( L(\cdot, \lambda^*) \).

- A particular case arises when \( L(\cdot, \lambda^*) \) has an unique minimizer. In this case the unique minimizer \( x^* \) of \( L \) is either primal feasible, and hence it is the primal-optimal solution, or it is not primal feasible, and then we can conclude that the no primal-optimal solution exists.
Example

Power allocation in a communication channel

We seek to best allocate a power level to $n$ communication channels. The problem can be formulated as

$$p^* = \min_x n \sum_{i=1}^n \log(\alpha_i + x_i) : x \geq 0, \sum_{i=1}^m x_i = 1.$$ 

where $\alpha_i > 0$ is a measure of the noise over the channel. Here the objective function is related to the communication rate. We use the Lagrangian

$$\mathcal{L}(x, \lambda, \nu) = -\sum_{i=1}^n \log(\alpha_i + x_i) - \lambda^\top x + \nu(\sum_{i=1}^m x_i - 1),$$

with $\lambda \in \mathbb{R}_+^n$, $\nu \in \mathbb{R}$.

\[\text{From Boyd & Vandenberghe's book, Convex Optimization.}\]
KKT conditions

Slater’s conditions are satisfied. The KKT conditions are:

- Primal feasibility: \( x \geq 0 \) and \( \mathbf{1}^T x = 1 \);
- Dual feasibility: \( \lambda \geq 0 \);
- Stationarity: \( \lambda_i + 1/(x_i + \alpha_i) = \nu, \ i = 1, \ldots, n \).
- Complementarity: \( \lambda_i x_i = 0, \ i = 1, \ldots, n \).

For an optimal pair \((x^*, \lambda^*, \nu^*)\):

- if \( \nu^* \leq 1/\alpha_i \), then \( 0 \leq \lambda_i^* \leq 1/\alpha_i - 1/(x_i^* + \alpha_i) = x_i/(\alpha_i(\alpha_i + x_i)) \). If \( \lambda_i^* > 0 \), then \( x_i^* = 0 \) from the complementarity conditions; this yields a contradiction. Hence \( \lambda_i = 0 \) and \( x_i^* = 1/\nu^* - \alpha_i(\geq 0) \) in that case.

- otherwise, \( \nu^* > 1/\alpha_i \); this leads to \( \lambda_i^* + 1/(x_i^* + \alpha_i) > 1/\alpha_i \). Again, assuming \( x_i^* > 0 \) leads to \( \lambda_i = 0 \) and a contradiction; hence \( x_i^* = 0 \) in that case.

We have obtained \( x_i^* = \max(0, 1/\nu^* - \alpha_i) \) for every \( i \). Summing, we obtain a condition that characterizes \( \nu^* \):

\[
1 = \sum_{i=1}^{n} x_i^* = \sum_{i=1}^{n} \max(0, 1/\nu^* - \alpha_i).
\]
Waterfilling algorithm

We can solve this 1D equation using a simple method called the waterfilling algorithm. Once $\nu^*$ is found, we then recover a primal optimal point via $x_i^* = \max(0, 1/\nu^* - \alpha_i)$, $i = 1, \ldots, n$.

The height of patch $i$ is given by $\alpha_i$. The region is flooded to a level $1/\nu$, using a total quantity of water equal to one. The height of the water (shown shaded) above each patch is the optimal value of $x_i$. 
Example

Maximum entropy distribution

Consider the problem

$$\min_x f_0(x) = \sum_{i=1}^{n} x_i \log x_i \quad : \quad x \geq 0, \quad 1^T x = 1.$$ 

The feasible set is the set of discrete distributions in $\mathbb{R}^n$; The objective function is called the negative entropy of the distribution $x$.

- Lagrangian: $L(x, \lambda, \nu) = f_0(x) - \lambda^T x + \nu(1 - 1^T x)$.
- KKT conditions: $x \geq 0, \quad 1^T x = 1, \quad \lambda \geq 0$, and

$$\lambda_i x_i = 0, \quad \log x_i = \lambda_i + \nu - 1, \quad i = 1, \ldots, n.$$ 

The stationarity conditions imply that $x^* > 0$, hence $\lambda^* = 0$, and thus $x_i$ does not depend on $i$. Since $1^T x = 1$, we obtain that $x^* = (1/n)1$, which is the uniform distribution.

This fact illustrates why the (negative) entropy function is used as a measure of “distance” between a distribution, to the uniform one.
**Example**

**Risk parity portfolio**

Consider a portfolio optimization problem: to find a portfolio weight vector $x \in \mathbb{R}^n_{++}$, containing positive dollar amounts to invest in various assets, such that the risk parity condition holds:

$$\forall i : x_i(Cx)_i = \frac{1}{n}x^\top Cx,$$

where $C = C^\top \succ 0$ is the (positive-definite) covariance of the assets. The interpretation of a risk-parity portfolio is that, since

$$\sum_{i=1}^{n} x_i(Cx)_i = x^\top Cx,$$

all the partial contributions $x_i(Cx)_i (> 0)$ of each asset $i$ to the total risk in the portfolio, as measured by its variance $x^\top Cx$, are equal (“at parity”).
Risk parity portfolio

Consider the optimization problem

$$\min_x f_0(x) + x^\top C x,$$

where

$$f_0(x) \equiv \begin{cases} -\sum_{i=1}^n \log x_i & \text{if } x > 0, \\ +\infty & \text{otherwise.} \end{cases}$$

Lagrangian:

$$\mathcal{L}(x, \lambda) = -\sum_{i=1}^n \log x_i + x^\top C x - \lambda^\top x.$$

KKT conditions: $x > 0$ (since $\mathcal{D} = \mathbb{R}^n_{++}$), $\lambda \geq 0$,

$$\lambda_i x_i = 0, \quad \frac{1}{x_i} + (C x)_i = \lambda_i, \quad i = 1, \ldots, n.$$

Since $x > 0$, we have $\lambda = 0$, and we obtain $x_i (C x)_i = 1$, $i = 1, \ldots, n$; summing, we get $x^\top C x = n$, which implies that the risk parity conditions hold.

This means that by solving the convex problem (7), we obtain a risk parity portfolio.